

# Numerical simulations of brittle and elasto-plastic fracture for thin structures subjected to dynamic loadings

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May 2012

# Introduction

- A thin body is a structure with a dimension largely smaller than the other ones
  - This dimension is called the thickness

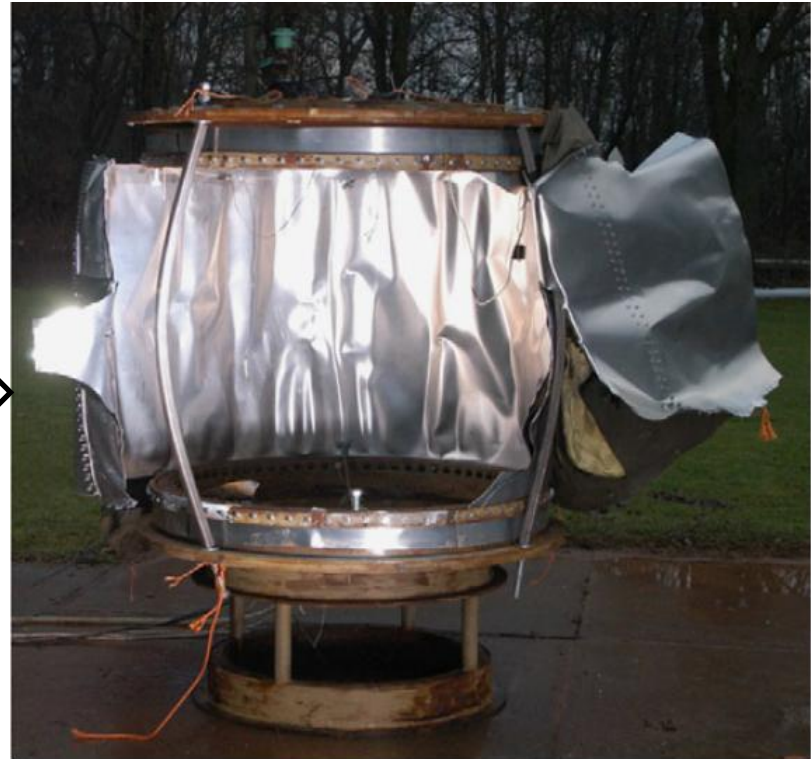


# Introduction

- Improve the safety of pressurized thin bodies by understanding their fracture behavior



Blast

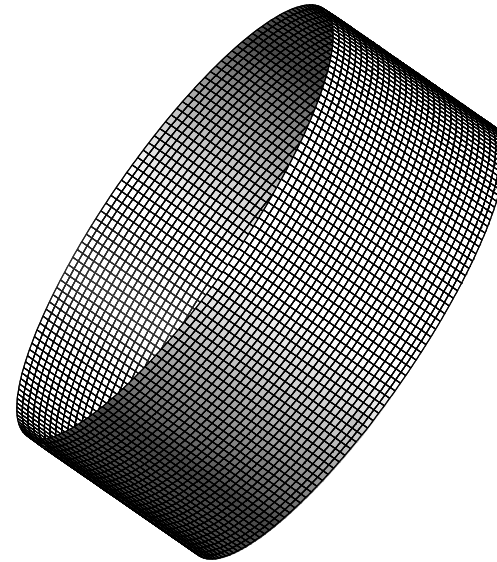
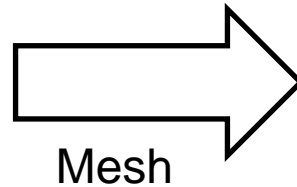


*[Larsson et al ijmme 2010]*

# Introduction

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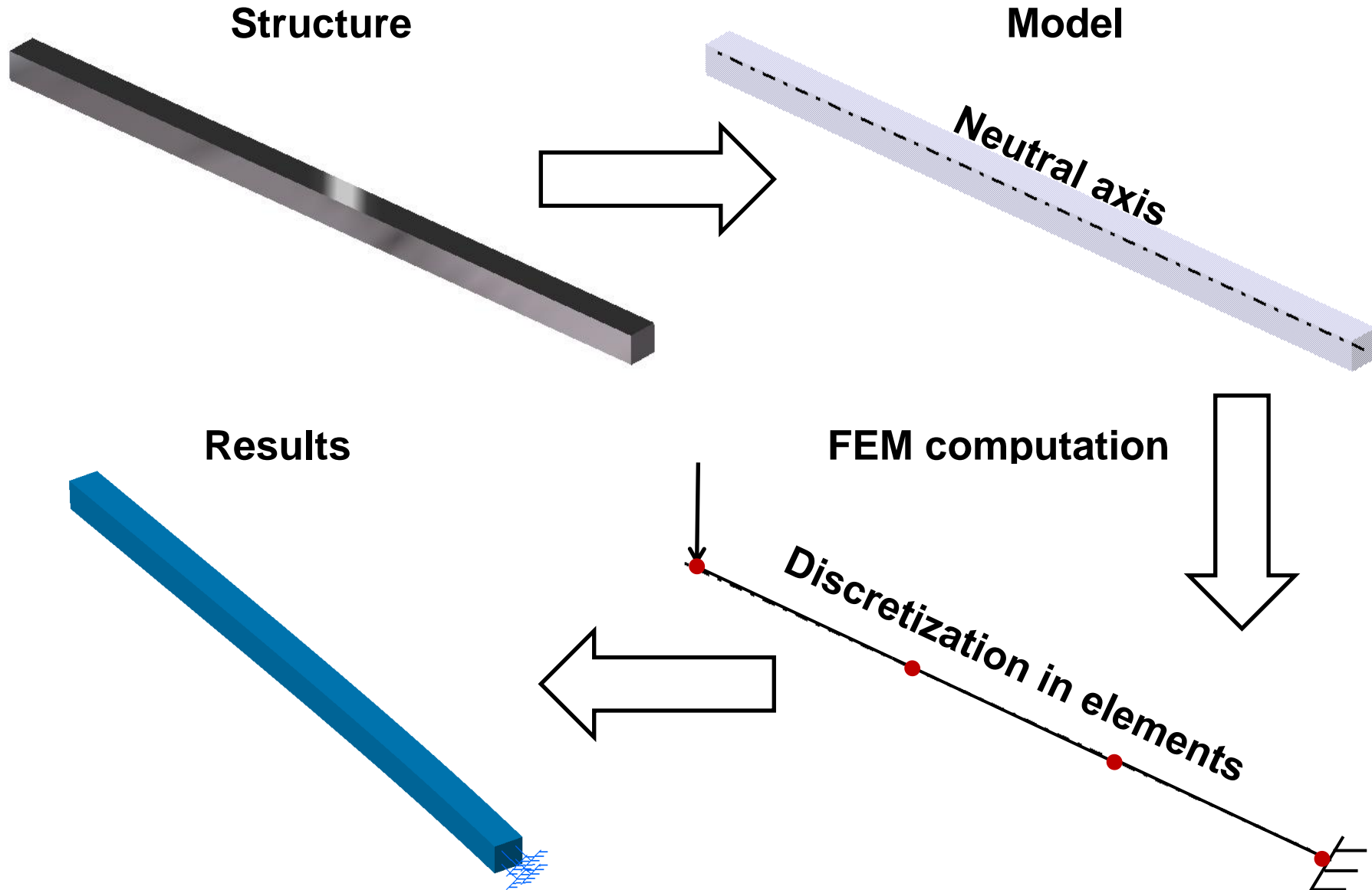
- Recourse to the finite element method allows cheaper designs
  - A numerical model is an idealization of reality based on mathematical equations
  - The finite element method (FEM) discretizes the structure in elements



*[Larsson et al ijnme 2010]*

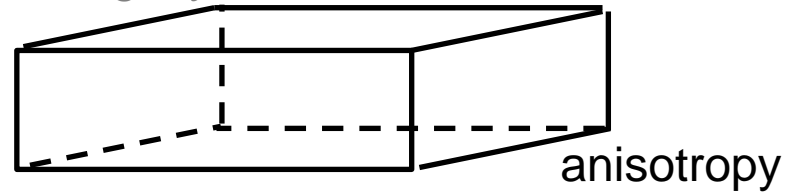
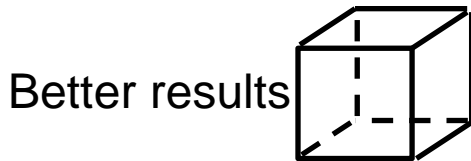
- The finite element method is a powerful tool in mechanics

# Introduction

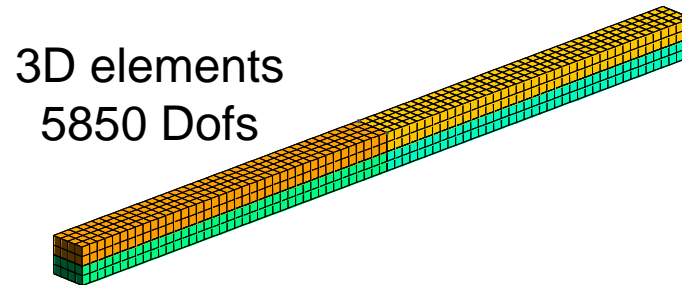


# Introduction

- Beam or shell elements can advantageously be used to model thin bodies
  - The dimensions of an element have to be slightly the same in all directions

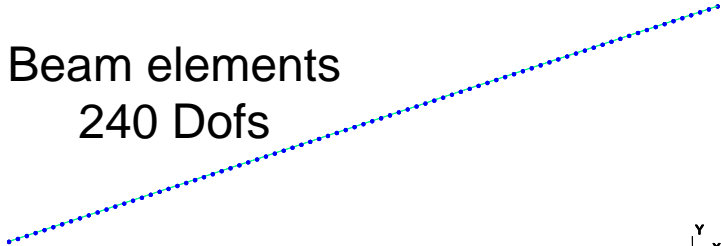


- Classical 3D elements leads to a huge number of elements for thin bodies

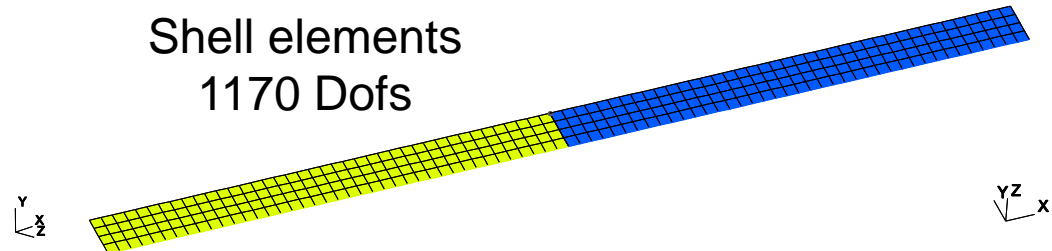


- Beam and shell elements use a 1D or 2D element as basis and account separately for the thickness → drastically reduces the time of computation

Beam elements  
240 Dofs

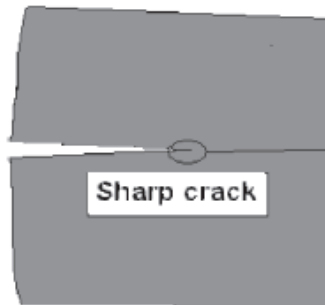


Shell elements  
1170 Dofs

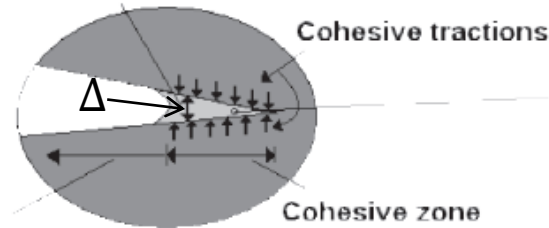


# Introduction

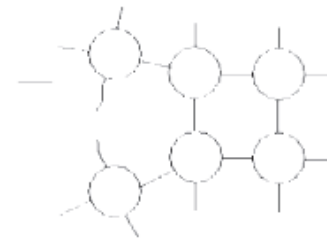
- Cohesive zone model is very appealing to model crack initiations in a numerical model
  - Model the separation of crack lips in brittle materials



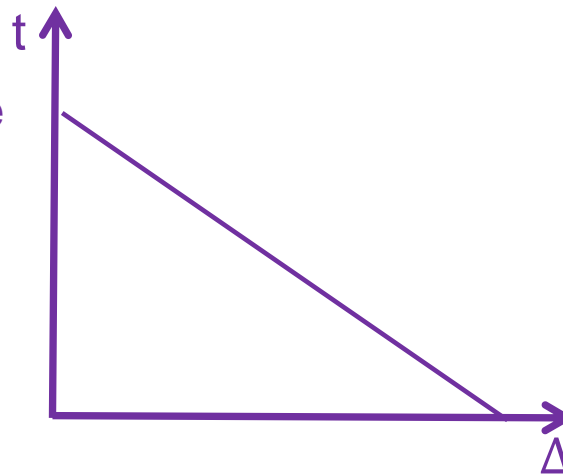
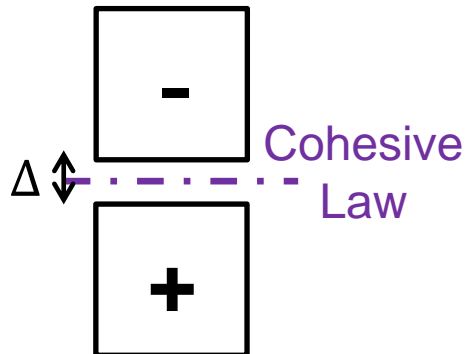
Crack face separation occurs across cohesive zone



Idealization of atomic separation processes in cohesive zone

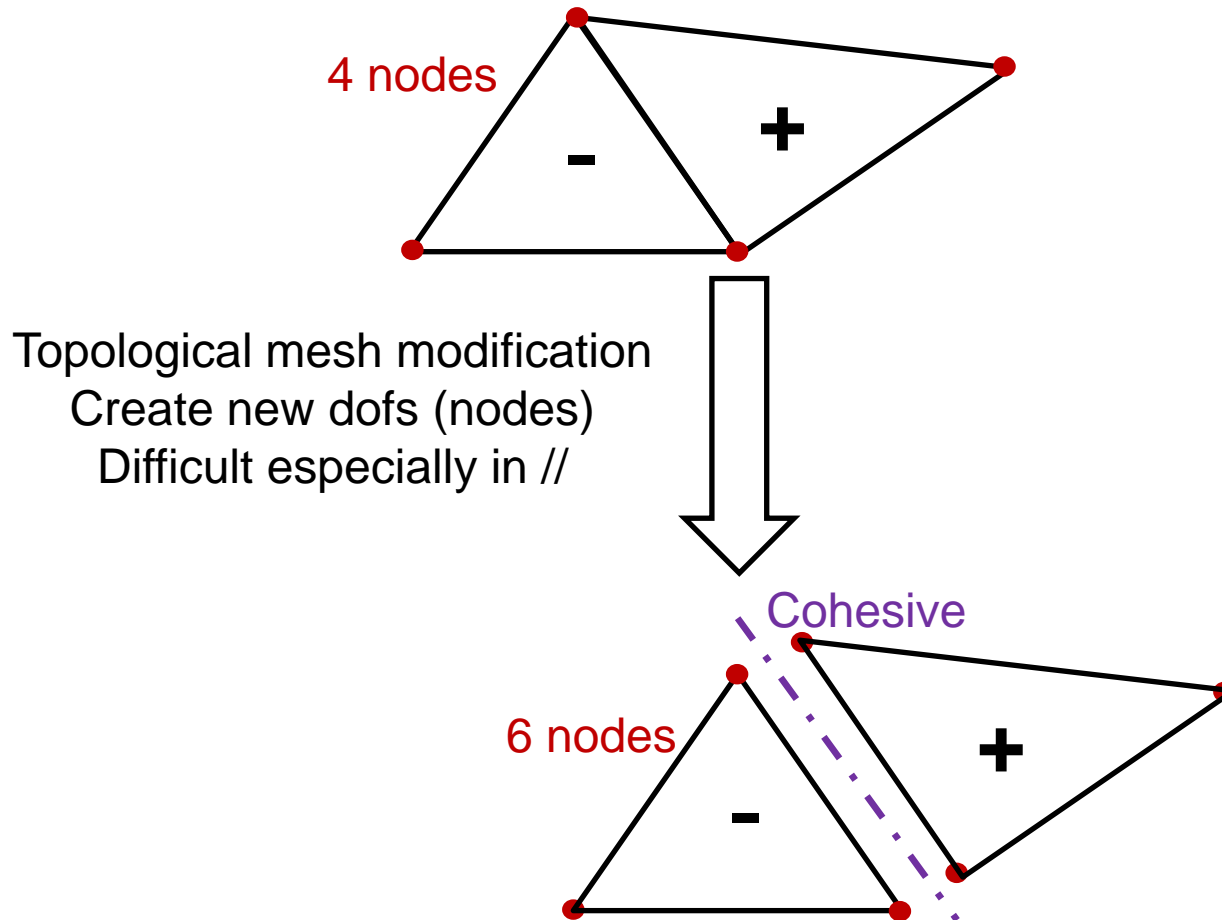


[Seagraves et al 2010]



# Introduction

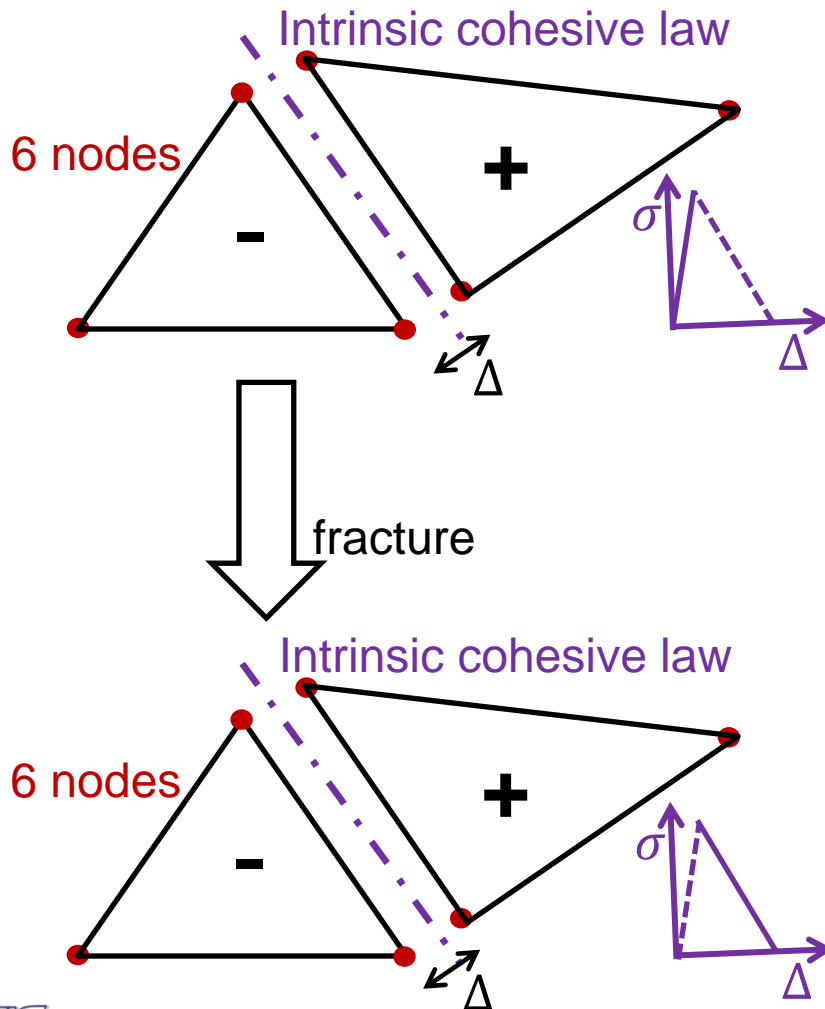
- The insertion of cohesive elements during the simulation is difficult to implement as it requires topological mesh modifications
  - FEM (continuous Galerkin)



# Introduction

- A recourse to an intrinsic cohesive law is generally done with FEM

– FEM (continuous Galerkin)

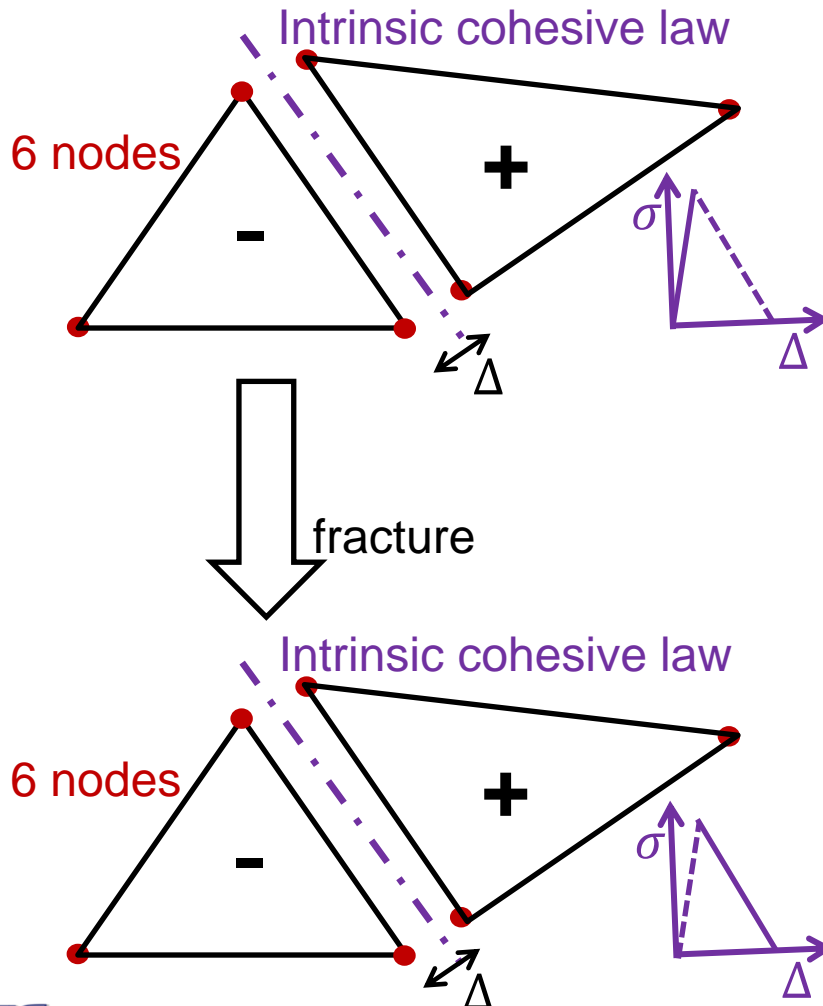


- Intrinsic cohesive law leads to numerical problems [*Seagraves et al 2010*]
  - Spurious stress wave propagation
  - Mesh dependency
  - Too fast crack propagation

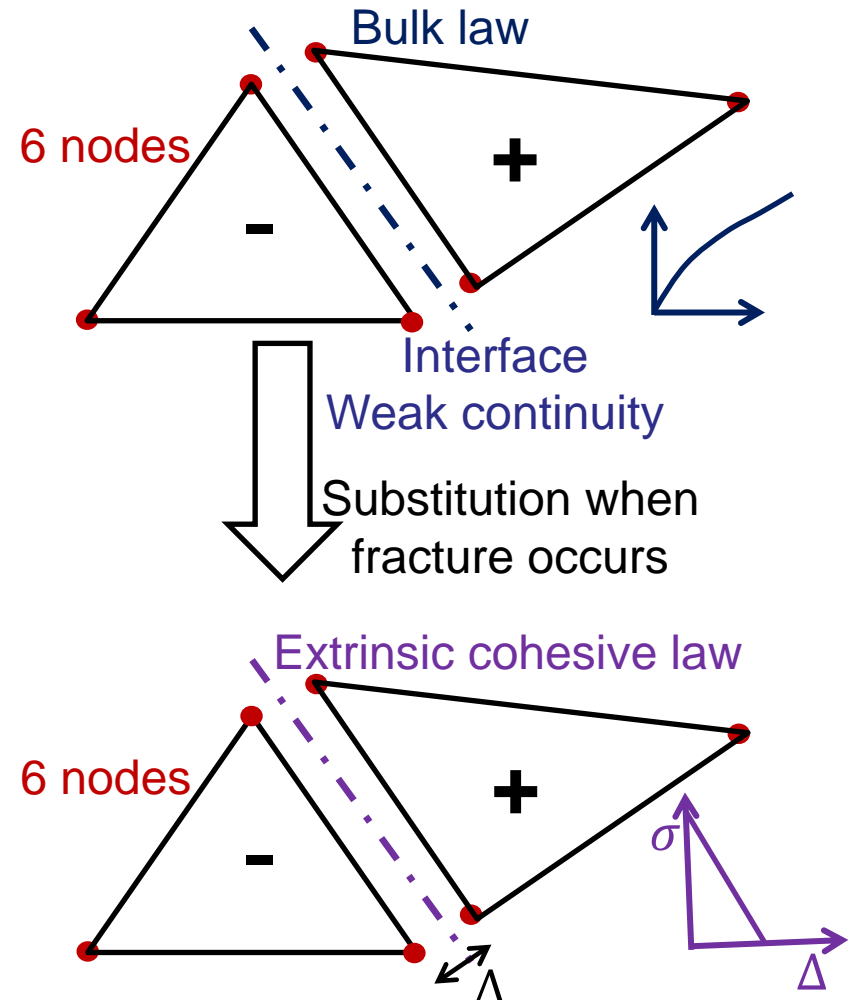
# Introduction

- Use of extrinsic cohesive law is easier when coupled with DG

– FEM (continuous Galerkin)



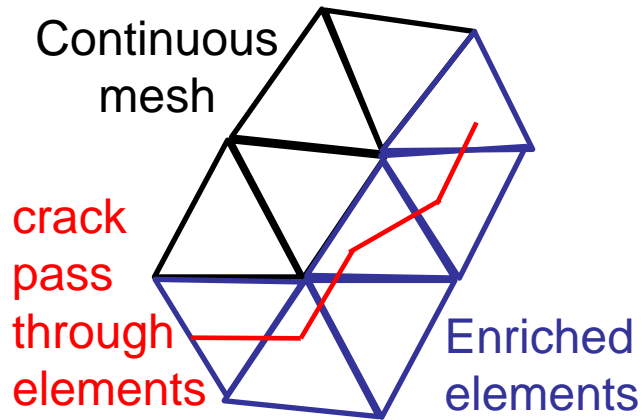
– Discontinuous Galerkin



# Introduction

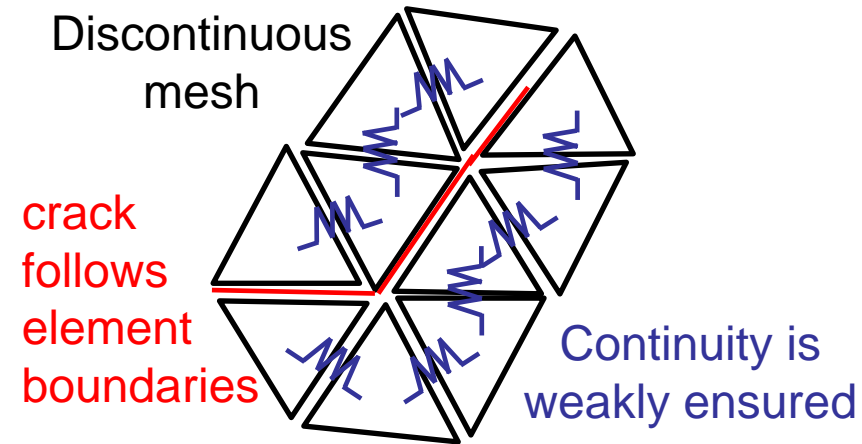
- Other methods exist but we focus on the discontinuous Galerkin method which has to be extended for thin bodies

## – XFEM



Commonly used for crack propagation

## – Discontinuous Galerkin



Recently developed for dynamic phenomena (crack propagation due to impact, fragmentation) but for 3D elements only

# Plan

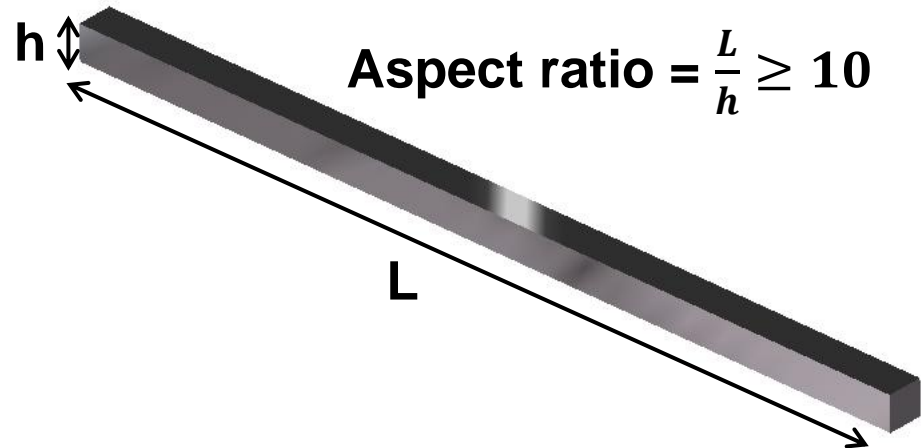
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- Develop a discontinuous Galerkin method for thin bodies
  - Beam elements (1.5D case)
  - Shell elements (2.5D case)
- Discontinuous Galerkin / Extrinsic Cohesive law framework
  - Develop a suitable cohesive law for thin bodies
- Applications
  - Fragmentations, crack propagations under blast loadings

# Full-DG formulation of Euler-Bernoulli beams

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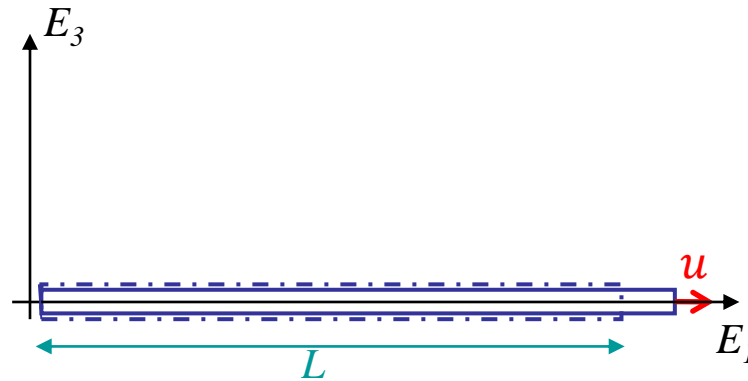
- Highlights
- Simple 1D thin structure
- Restrict the analysis to
  - Linear small strains
  - Straight rectangular beam (without initial deformation)
  - Out-of-plane shearing can be neglected
  - Plane stress state



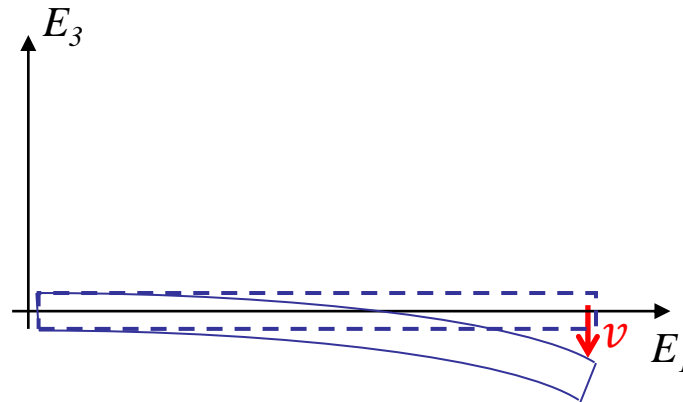
# Full-DG formulation of Euler-Bernoulli beams

- 2 (independent in small deformations) deformation modes (shearing is neglected)

– Membrane mode



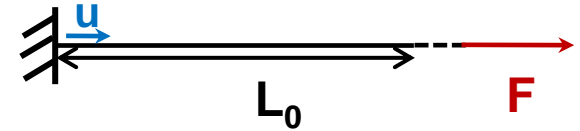
– Bending mode



# Full-DG formulation of Euler-Bernoulli beams

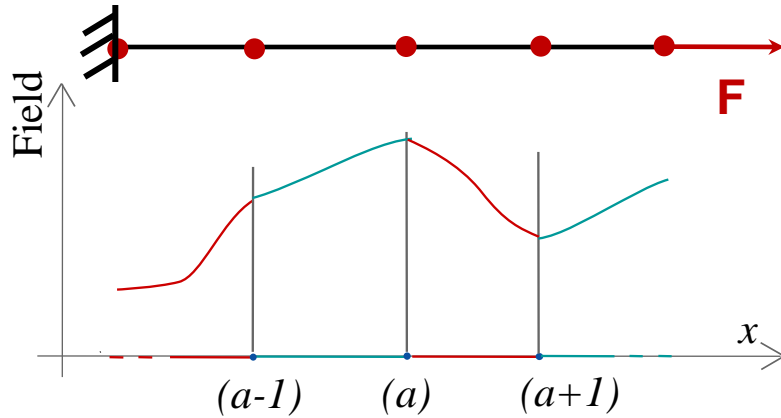
- Membrane mode

- Strong form  $(n^{11})_{,1} = 0$  with  $n^{11} = \int_{-h/2}^{h/2} \sigma^{11} d\xi^3$



- Weak form  $\int_0^L (n^{11})_{,1} \delta u dx = 0$

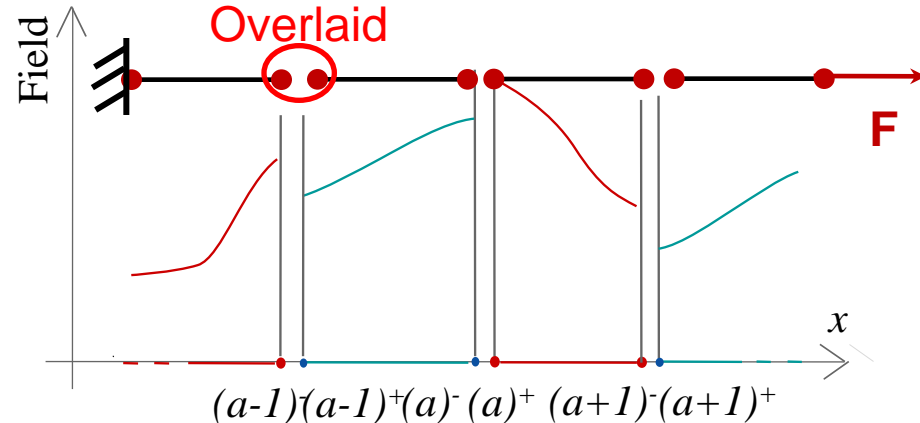
- FEM (Continuous Galerkin)



Integration by parts on the beam

$$\sum_e \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}} = 0$$

- Discontinuous Galerkin

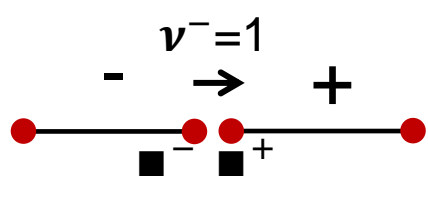


Integration by parts on each element

$$\sum_e \left( \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}} - \underbrace{n^{11} \delta u}_{\text{Interface term}} \Big|_{l_e} \right) = 0$$

- The interface terms are developed

- Operators definition



The diagram shows a horizontal line representing a beam element. There are four red dots representing nodes. The leftmost and rightmost dots are connected by a solid line. Between the two inner dots, there is a gap representing an interface. Above the line, a minus sign is on the left and a plus sign is on the right, with an arrow pointing from the minus to the plus. Below the line, a square with a minus sign is on the left and a square with a plus sign is on the right, positioned at the interface. Above the line, the text  $v^- = 1$  is written.

Mean:  $\langle \blacksquare \rangle = \frac{1}{2} (\blacksquare^+ + \blacksquare^-)$

Jump:  $[[ \blacksquare ]] = \blacksquare^+ - \blacksquare^-$

- Using operators

$$-\sum_e n^{11} \delta u]_{l_e} = \sum_s [[n^{11} \delta u]]_s$$

- Using mathematical identity  $[[ab]] = \langle a \rangle [[b]] + [[a]] \langle b \rangle$

$$-\sum_e n^{11} \delta u]_{l_e} = \sum_s (\langle n^{11} \rangle [[\delta u]] + [[n^{11}]] \langle \delta u \rangle)_s$$

# Full-DG formulation of Euler-Bernoulli beams

- The jump is replaced by a consistent numerical flux (no equality)

$$-\sum_e n^{11} \delta u|_{l_e} = \sum_s [[n^{11} \delta u]]_s = \sum_s (\langle n^{11} \rangle [[\delta u]] + \underbrace{[[n^{11}]] \langle \delta u \rangle}_0)_s \neq \sum_s (\langle n^{11} \rangle [[\delta u]])_s$$

for the exact continuous solution (consistency is preserved)

- Main idea of DG

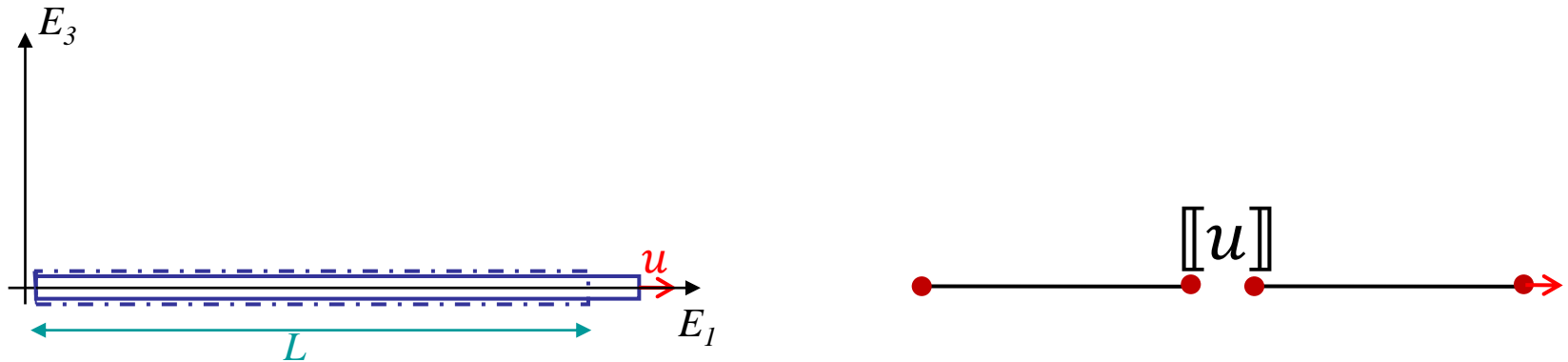
- The governing equation becomes

$$\sum_e \left( \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}} - \underbrace{n^{11} \delta u|_{l_e}}_{\text{Interface term}} \right) \neq \sum_e \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}} + \sum_s \underbrace{(\langle n^{11} \rangle [[\delta u]])_s}_{\text{Consistency term}} = 0$$

- ≠ pure penalty method (Intrinsic cohesive law) which does not include the consistency terms

# Full-DG formulation of Euler-Bernoulli beams

- Discontinuous elements  $\rightarrow$  displacement jumps have to be constrained
  - Membrane mode



- Continuity is weakly ensured by symmetrization terms

$$\sum_s (\langle Eh\delta u_{,1} \rangle [[u]])_s = 0$$

for the exact continuous solution  
 $\rightarrow$  consistency is preserved

- Method is stabilized by quadratic terms

$$\left. \begin{array}{l} \sum_s (\langle n^{11} \rangle \llbracket \delta u \rrbracket)_s \\ \sum_s (\langle Eh \delta u_{,1} \rangle \llbracket u \rrbracket)_s \end{array} \right\} \rightarrow \sum_s \left( \begin{array}{c} \llbracket u \rrbracket \\ \parallel \\ 0 \end{array} \left\langle \frac{Eh\beta_2}{h^s} \right\rangle \llbracket \delta u \rrbracket \right)_s = 0$$

for the exact continuous solution  
→ consistency is preserved

- $\beta_2 > 1$  dimensionless stability parameter (Practically stable if  $\beta_2 \geq 10$ )
- $h^s$  characteristic mesh size which ensures the dimensionless nature of  $\beta_2$

# Full-DG formulation of Euler-Bernoulli beams

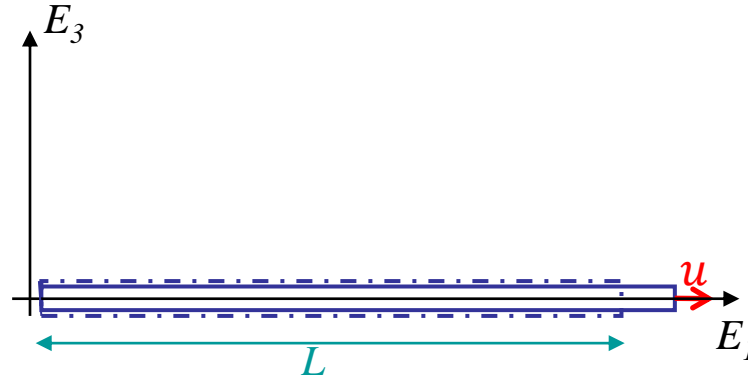
- The final equation (membrane mode) is obtained by adding the terms

$$\sum_e \left( \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}} - \underbrace{n^{11} \delta u}_{l_e} \right) \rightarrow \boxed{\sum_e \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}}} + \sum_s \left( \underbrace{\langle n^{11} \rangle \llbracket \delta u \rrbracket}_{\text{Consistency term}} \right)_s + \sum_s \left( \underbrace{\langle Eh \delta u_{,1} \rangle \llbracket u \rrbracket}_{\text{Symmetrization term}} \right)_s + \sum_s \left( \underbrace{\llbracket u \rrbracket \left\langle \frac{Eh\beta_2}{h^s} \right\rangle \llbracket \delta u \rrbracket}_{\text{Stability term}} \right)_s = 0$$

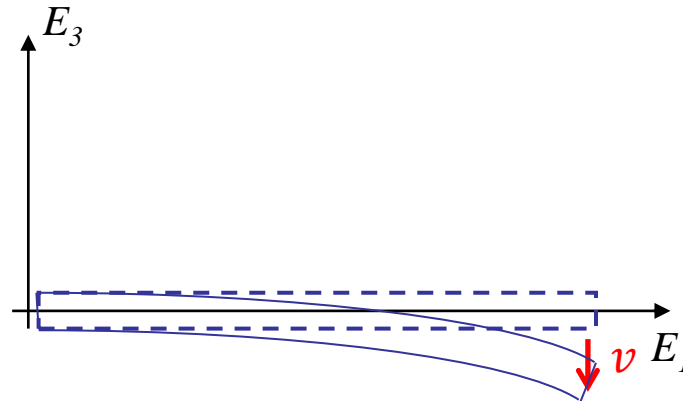
- Consistent, (Weakly) continuous and stable
- Same as FEM but with extra interface terms

# Full-DG formulation of Euler-Bernoulli beams

- 2 (independent) deformation modes (shearing is neglected)
  - Membrane mode (OK)

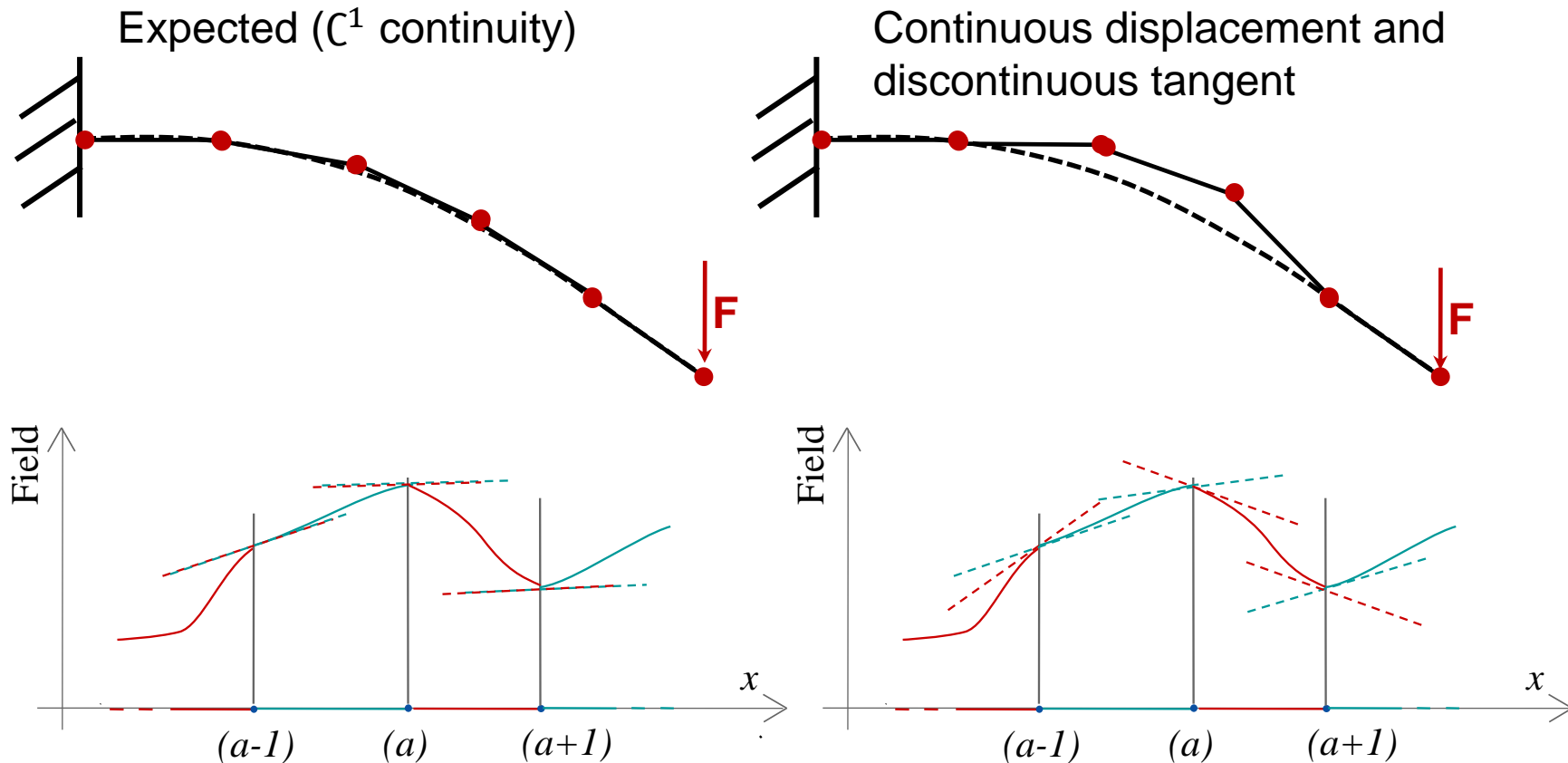


- Bending mode



# Full-DG formulation of Euler-Bernoulli beams

- The bending mode requires the  $C^1$  continuity (*i.e.* the tangent continuity)
  - For FEM without rotational Dofs



# Full-DG formulation of Euler-Bernoulli beams

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- Several techniques exist to ensure the tangent continuity using FEM
  - $C^1$  Shape functions (beams only)
  - Recourse to rotational degrees of freedom (2-field formulation)
  - Lagrange multipliers (add degrees of freedom)
  - ...

## Full-DG formulation of Euler-Bernoulli beams

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- The discontinuous Galerkin method can be advantageously used to ensure the tangent continuity
  - Ensured weakly by interface terms
  - $C^0$ /DG method (elements are continuous)
  - One-field formulation (displacements are the only unknowns)
  - First DG methods for thin bodies formulation [*Engel et al cmame 2002*]

# Full-DG formulation of Euler-Bernoulli beams

- The form of the DG formulation is similar to the one obtained for the membrane problem
  - Strong form  $(m^{11})_{,1} = 0$  with  $m^{11} = \int_{-h/2}^{h/2} \sigma^{11} \xi^3 d\xi^3$
  - Weak form  $\int_L (m^{11})_{,1} \delta(-v_{,1}) dx = 0$
  - Shearing is neglected
  - External forces and inertial parts are omitted
  - FEM (Continuous Galerkin)                      – Discontinuous Galerkin

$$\sum_e \int_{l_e} \underbrace{m^{11} \delta(-v_{,1}) dx}_{\text{Bulk term}} = 0 \qquad \sum_e \left( \int_{l_e} \underbrace{m^{11} \delta(-v_{,1}) dx}_{\text{Bulk term}} - \underbrace{m^{11} \delta(-v_{,1}) \Big|_{l_e}}_{\text{Interface term}} \right) = 0$$

# Full-DG formulation of Euler-Bernoulli beams

- 3 interfaces terms are considered following the framework made for the membrane mode

- Consistent terms

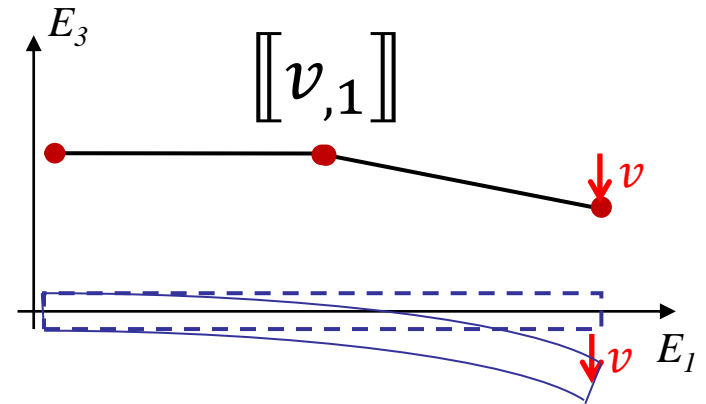
$$-\sum_e m^{11} \delta(-v_{,1}) \llbracket \rrbracket_{l_e} = \sum_s \llbracket m^{11} \delta(-v_{,1}) \rrbracket_s \rightarrow \sum_s (\langle m^{11} \rangle \llbracket \delta(-v_{,1}) \rrbracket)_s$$

- Symmetrization terms

$$\sum_s \left( \left\langle \frac{Eh^3}{12} \delta(-v_{,11}) \right\rangle \llbracket -v_{,1} \rrbracket \right)_s = 0$$

- Stability terms

$$\sum_s \left( \llbracket -v_{,1} \rrbracket \left\langle \frac{Eh^3 \beta_1}{12h^s} \right\rangle \llbracket \delta(-v_{,1}) \rrbracket \right)_s = 0$$



$\beta_1 > 1$  dimensionless stability parameter

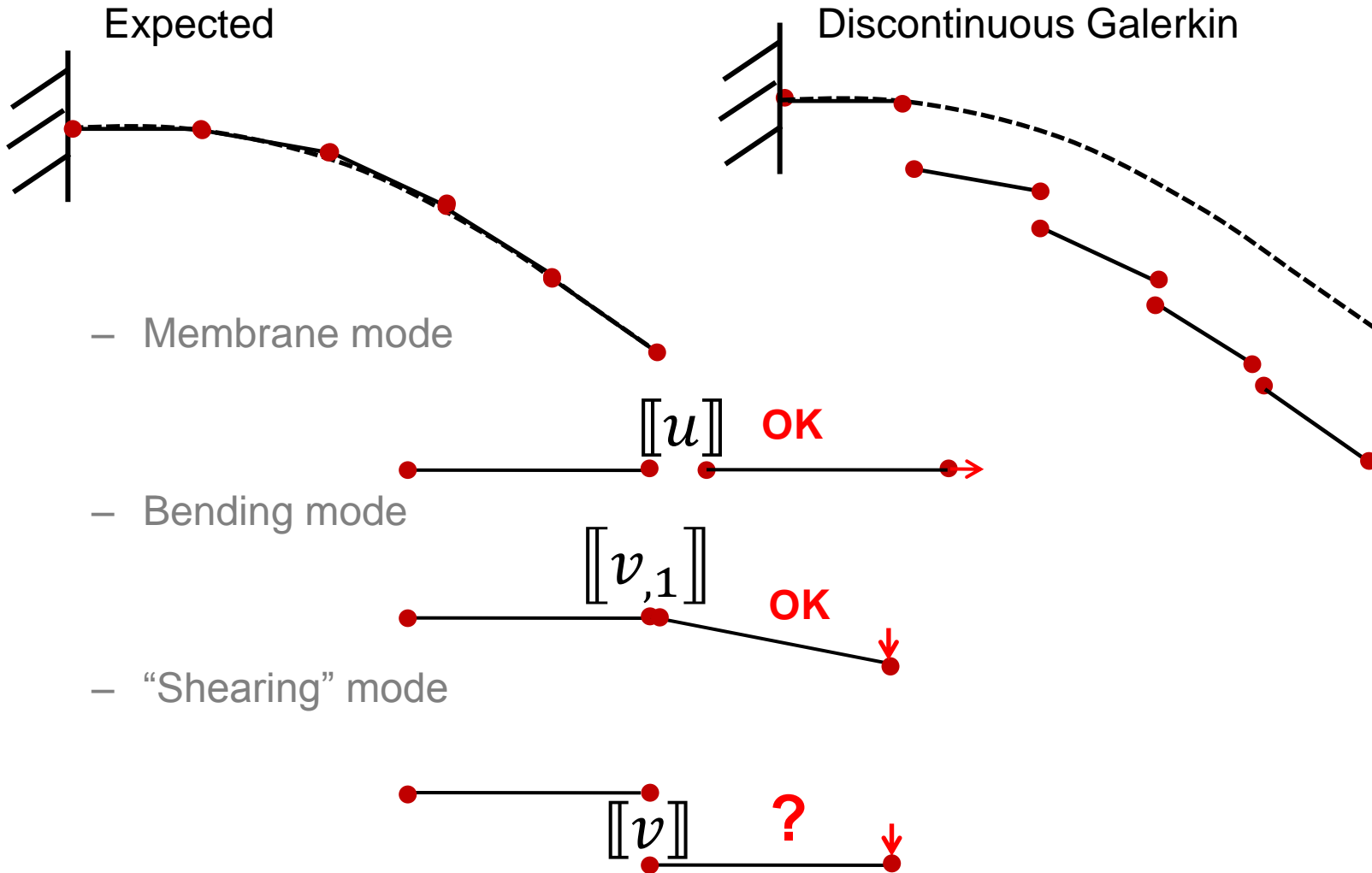
- Bending equation

$$\begin{aligned}
 \sum_e \left( \int_{l_e} \underbrace{m^{11} \delta(-v_{,11}) dx}_{\text{Bulk term}} - \underbrace{m^{11} \delta(-v_{,1}) \Big|_{l_e}}_{\text{Interface term}} \right) &\rightarrow \sum_e \int_{l_e} \underbrace{m^{11} \delta(-v_{,11}) dx}_{\text{Bulk term}} + \\
 &\sum_s \left( \underbrace{\langle m^{11} \rangle \llbracket \delta(-v_{,1}) \rrbracket}_{\text{Consistency term}} \right)_s + \\
 &\sum_s \left( \underbrace{\left\langle \left( \frac{Eh^3}{12} \right) \delta(-v_{,11}) \right\rangle \llbracket -v_{,1} \rrbracket}_{\text{Symmetrization term}} \right)_s + \\
 &\sum_s \left( \underbrace{\llbracket -v_{,1} \rrbracket \left\langle \frac{Eh^3 \beta_1}{12h^s} \right\rangle \llbracket \delta(-v_{,1}) \rrbracket}_{\text{Stability term}} \right)_s = 0
 \end{aligned}$$

- Consistent, stable and weakly continuous thanks to interface terms
- Same as FEM with extra interface terms

# Full-DG formulation of Euler-Bernoulli beams

- Out-of-plane continuity is not ensured



# Full-DG formulation of Euler-Bernoulli beams

- Out-of-plane continuity is ensured by introducing an interface term in  $\delta v$ 
  - Account (temporarily) for negligible shearing in the simplified bending equation

$$\int_L [(m^{11})_{,1} \delta(-v_{,1}) - l^1 \delta(-v_{,1})] dx = 0 \quad \text{with} \quad l^1 = \int_{-h/2}^{h/2} \sigma^{31} d\xi^3 \approx 0$$

Simplified bending equation

- Unusual integration by parts on  $\delta(-v_{,1})$  for the shearing term

Term in  $\delta v_{,1}$  to constrain  $[[v_{,1}]]$

$$\int_L [(m^{11})_{,1} \delta(-v_{,1}) - l^1 \delta(-v_{,1})] dx = \sum_e \left( \int_{l_e} \underbrace{m^{11} \delta(-v_{,11}) dx}_{\text{Bulk term}} - \underbrace{m^{11} \delta(-v_{,1})}_{\text{Interface term}} \right)_{l_e} - \left( \int_{l_e} \underbrace{(l^1)_{,1} \delta(-v) dx}_{\text{Bulk term}} + \underbrace{l^1 \delta(-v)}_{\text{Interface term}} \right)_{l_e} = 0$$

Term in  $\delta v \rightarrow$  We can ensure weakly this continuity using DG

# Full-DG formulation of Euler-Bernoulli beams

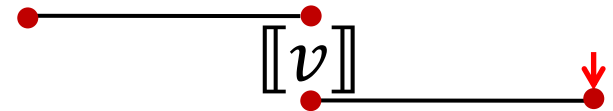
- 3 interface terms are derived from  $l^1 \delta(-v)]_{l_e}$  exactly as for the membrane and bending modes

- Consistency terms

$$\sum_e l^1 \delta(-v)]_e = - \sum_s \llbracket l^1 \delta(-v) \rrbracket_s \rightarrow - \sum_s (\langle l^1 \rangle \llbracket \delta(-v) \rrbracket)_s$$

- Symmetrization terms

$$\sum_s \left( \left\langle \frac{Eh}{2(1+\nu)} \delta(-v,1) \right\rangle \llbracket -v \rrbracket \right)_s = 0$$



- Stability terms

$$\sum_s \left( \llbracket -v \rrbracket \left\langle \frac{Eh\beta_3}{2(1+\nu)} \right\rangle \llbracket \delta(-v) \rrbracket \right)_s = 0 \quad \beta_3 > 0 \text{ dimensionless stability parameter}$$

# Full-DG formulation of Euler-Bernoulli beams

- Only the stabilization terms remain as the shearing is neglected (Euler-Bernoulli assumption)

- Consistency terms

$$\sum_e l^1 \delta(-v)]_e = - \sum_s \llbracket l^1 \delta(-v) \rrbracket_s \rightarrow - \sum_s (\langle l^1 \rangle \llbracket \delta(-v) \rrbracket)_s \approx 0 \quad \text{NEGLECTED}$$

- Symmetrization terms

$$\sum_s \left( \left\langle \frac{Eh}{2(1+\nu)} \delta(-v,1) \right\rangle \llbracket -v \rrbracket \right)_s = 0 \quad \text{ENSURES CONTINUITY BUT LEAD TO UNSYMMETRIC FORMULATION} \\ \rightarrow \text{IS NOT CONSIDERED}$$

- Stability terms

$$\sum_s \left( \llbracket -v \rrbracket \left\langle \frac{Eh\beta_3}{2(1+\nu)h^s} \right\rangle \llbracket \delta(-v) \rrbracket \right)_s = 0 \quad \text{ENSURES STABILTY AND CONTINUITY}$$

# Full-DG formulation of Euler-Bernoulli beams

- The final full-DG equation is obtained by adding the different contributions (membrane + bending) [*Becker et al , ijnme 2011*]

$$\sum_e \int_{l_e} \underbrace{n^{11} \delta u_{,1} dx}_{\text{Bulk term}} + \sum_e \int_{l_e} \underbrace{m^{11} \delta(-v_{,11}) dx}_{\text{Bulk term}} +$$

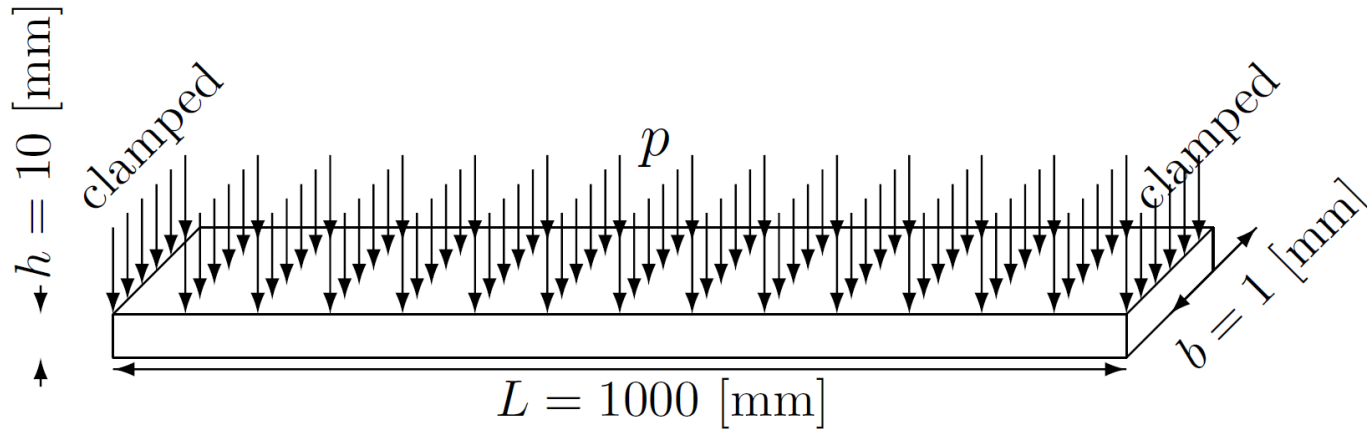
FEM (CG) equation

$$\begin{aligned} & \sum_s \left( \underbrace{\langle n^{11} \rangle \llbracket \delta u \rrbracket}_{\text{Consistency term}} \right)_s + \sum_s \left( \underbrace{\langle Eh \delta u_{,1} \rangle \llbracket u \rrbracket}_{\text{Symmetrization term}} \right)_s + \sum_s \left( \underbrace{\llbracket u \rrbracket \left\langle \frac{Eh\beta_2}{h^s} \right\rangle \llbracket \delta u \rrbracket}_{\text{Stability term}} \right)_s \\ & \sum_s \left( \underbrace{\langle m^{11} \rangle \llbracket \delta(-v_{,1}) \rrbracket}_{\text{Consistency term}} \right)_s + \sum_s \left( \underbrace{\left\langle \left( \frac{Eh^3}{12} \right) \delta(-v_{,11}) \right\rangle \llbracket -v_{,1} \rrbracket}_{\text{Symmetrization term}} \right)_s + \sum_s \left( \underbrace{\llbracket -v_{,1} \rrbracket \left\langle \frac{Eh^3\beta_1}{12h^s} \right\rangle \llbracket \delta(-v_{,1}) \rrbracket}_{\text{Stability term}} \right)_s \\ & + \sum_s \left( \underbrace{\llbracket -v \rrbracket \left\langle \frac{Eh\beta_3}{2(1+\nu)h^s} \right\rangle \llbracket \delta(-v) \rrbracket}_{\text{Stability term}} \right)_s \end{aligned}$$

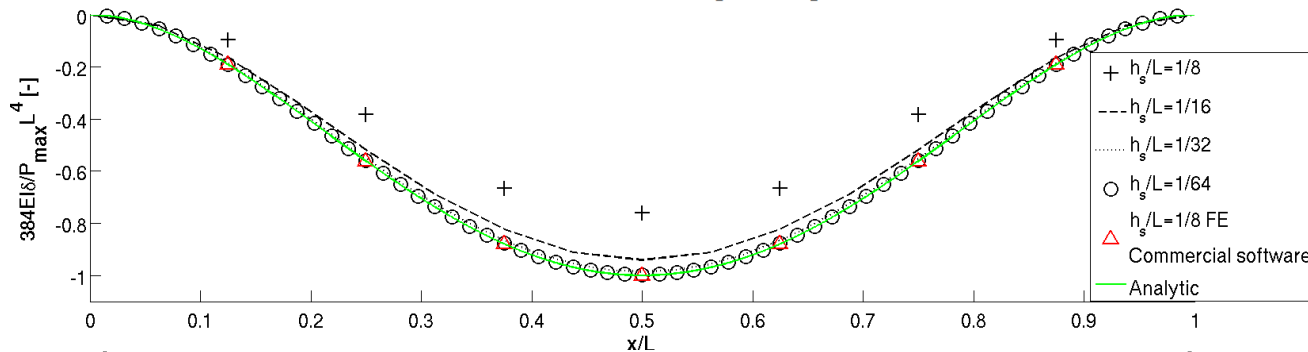
= 0

# Full-DG formulation of Euler-Bernoulli beams

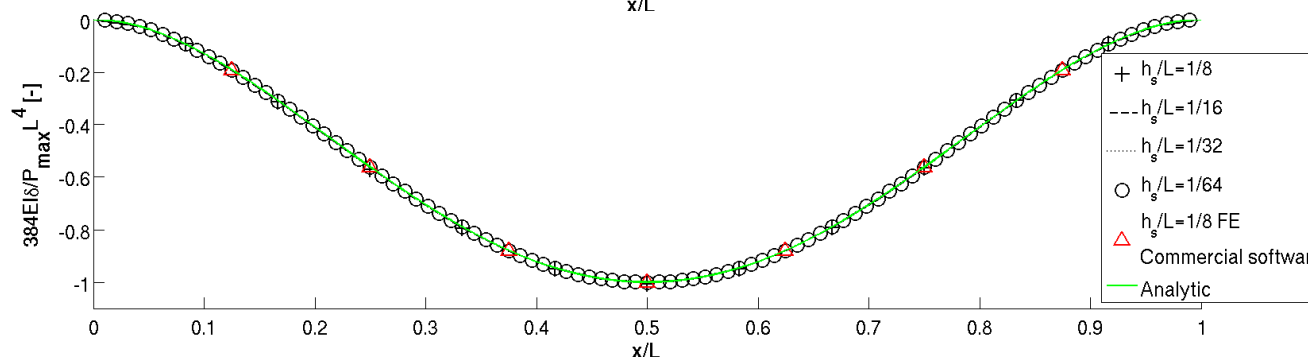
- The analytical solution is matched with discontinuous elements



2<sup>nd</sup> order elements

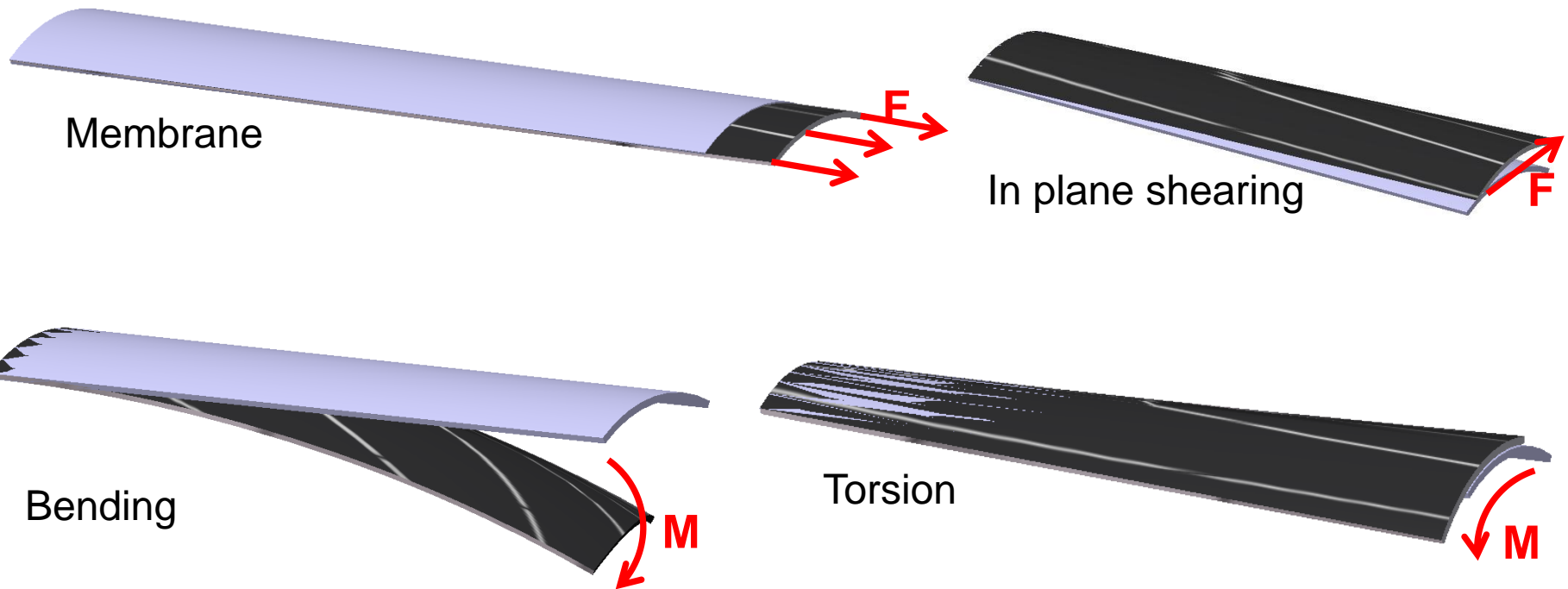


3<sup>rd</sup> order elements



# Full-DG formulation of Kirchhoff-Love shells

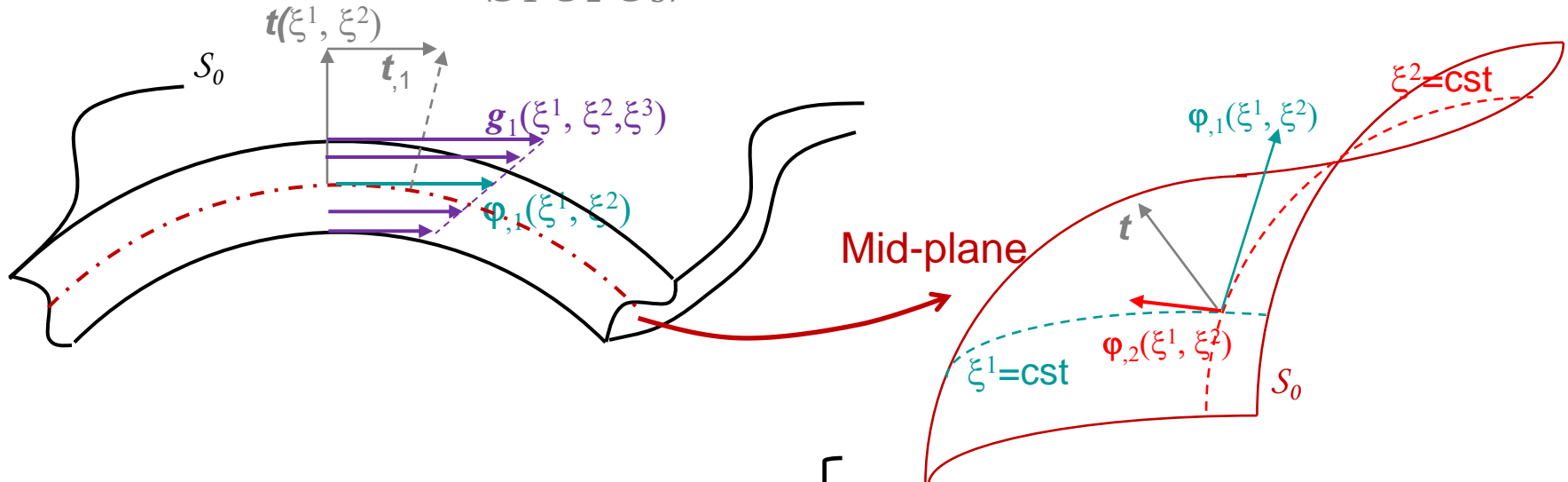
- Structure whose thickness is  $\ll$  other dimensions
- Initial curvature (otherwise it is a plate)  $\Leftrightarrow$  bending/membrane coupling
- Modes
  - Out-of-plane shearing is neglected (Kirchhoff-Love theory)



# Full-DG formulation of Kirchhoff-Love shells

- The kinematics of the shell is formulated in a basis linked to the shell

- Convector basis  $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$



$$\mathbf{g}_\alpha = \boldsymbol{\varphi}_{,\alpha} + \xi^3 \lambda_h \mathbf{t}_{,\alpha}$$

$$\mathbf{g}_3 = \lambda_h \mathbf{t}$$

where  $\left\{ \begin{array}{l} \lambda_h \text{ accounts for a thickness variation} \\ \mathbf{t} = \mathbf{t}(\xi^1, \xi^2, \xi^3) \text{ by assumption} \end{array} \right.$

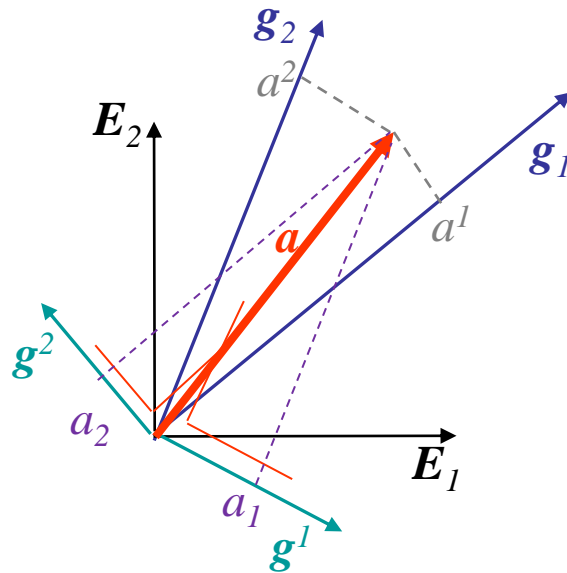
- The convected basis is not orthonormal
- Curvature of the shell is characterized by

$$\lambda_\alpha^\beta = \mathbf{t}_{,\alpha} \cdot \boldsymbol{\varphi}^{\beta}$$

# Full-DG formulation of Kirchhoff-Love shells

- As the convected basis is not orthonormal, a conjugate (or dual) basis is defined to decompose vectors or matrices

$$\mathbf{g}_I \cdot \mathbf{g}^J = \delta_{IJ}$$



The vector  $\mathbf{a}$  can be formulated in both bases:

$$\mathbf{a} = a^1 \mathbf{g}_1 + a^2 \mathbf{g}_2$$

$$\mathbf{a} = a_1 \mathbf{g}^1 + a_2 \mathbf{g}^2$$

And (for example)

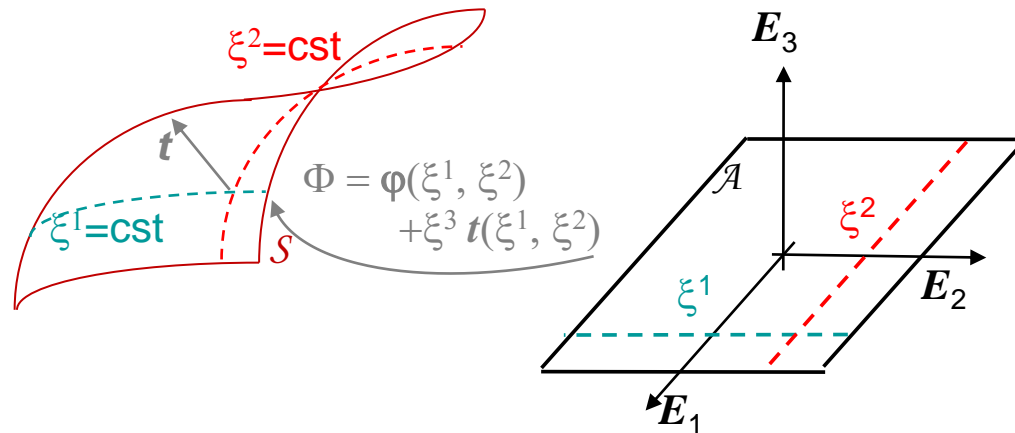
$$\mathbf{a} \cdot \mathbf{g}^1 = (a^1 \mathbf{g}_1 + a^2 \mathbf{g}_2) \cdot \mathbf{g}^1 = a^1$$

# Full-DG formulation of Kirchhoff-Love shells

- The equations are formulated in the reference frame
  - The Jacobian describes the change between the configurations

Shell configuration

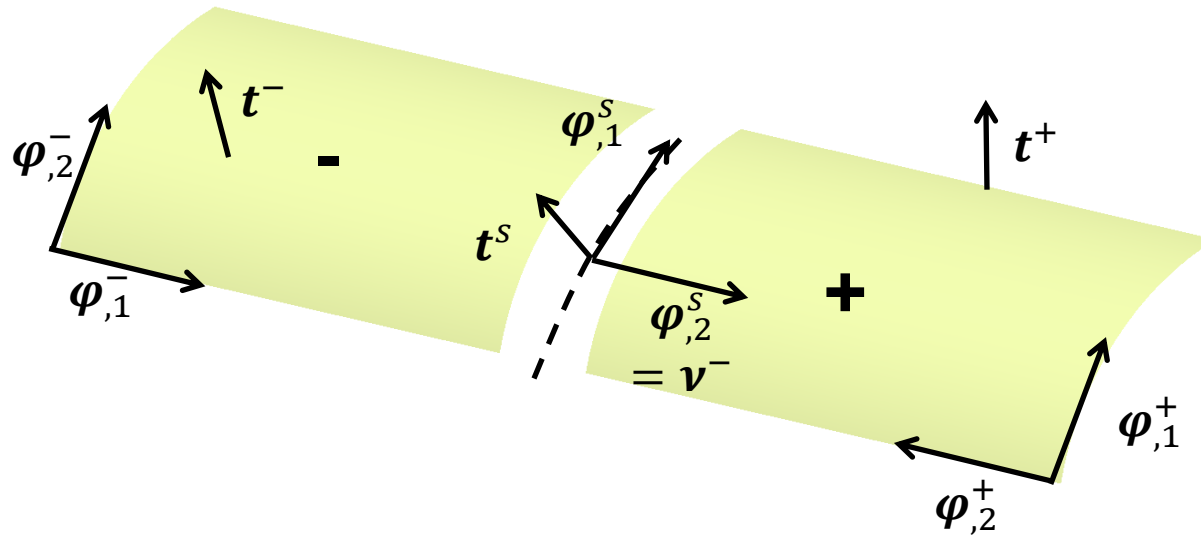
Reference configuration



$$j = \det(\nabla\Phi) = (\mathbf{g}_1 \wedge \mathbf{g}_2) \cdot \mathbf{g}_3$$
$$\bar{j} = \lambda_h(\boldsymbol{\varphi}_{,1} \wedge \boldsymbol{\varphi}_{,2}) \cdot \mathbf{t}$$

# Full-DG formulation of Kirchhoff-Love shells

- The normal at the interface is chosen as the outward normal to the minus element (convention)



– Normal components

$$v_{\alpha}^{-} = \varphi_{,2}^s \cdot \varphi_{,\alpha}^s$$

# Full-DG formulation of Kirchhoff-Love shells

- The stress tensor  $\sigma$  is integrated on the thickness in the convected basis

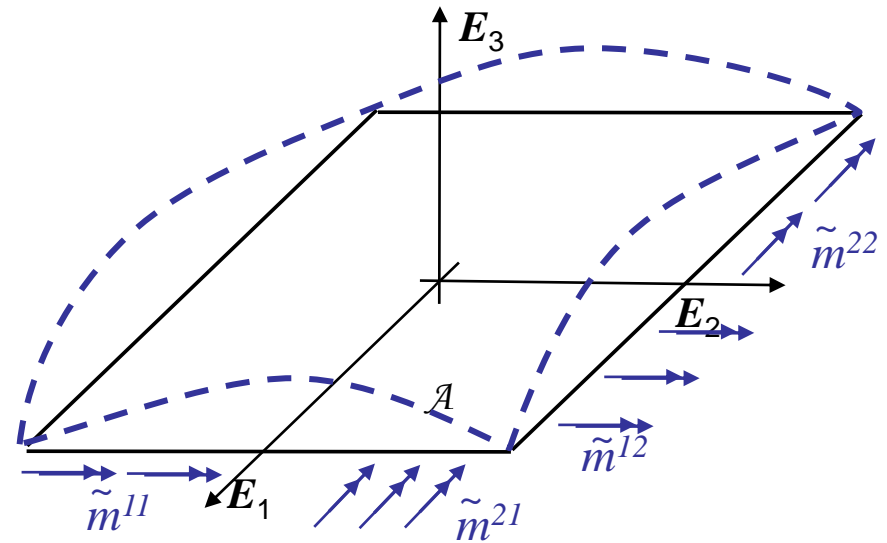
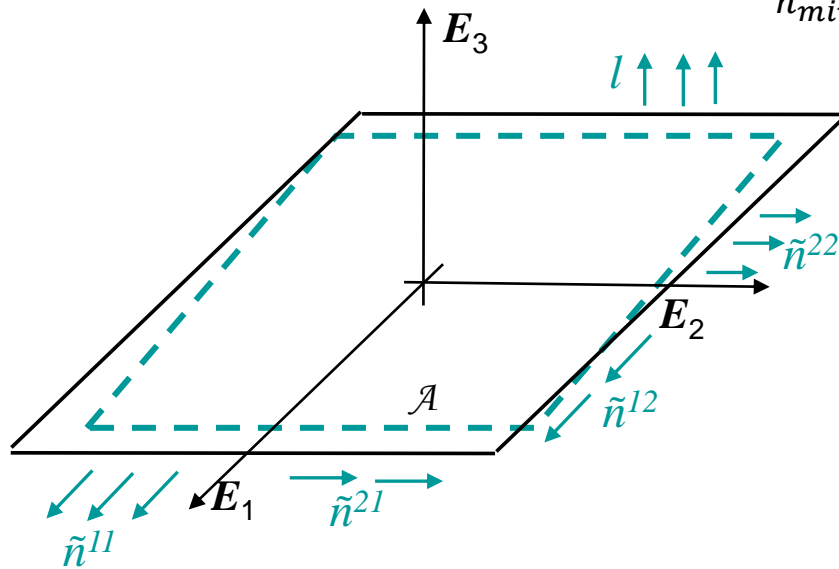
– Reduced stresses

$$\mathbf{n}^\alpha = \frac{1}{\bar{j}} \int_{h_{min}}^{h_{max}} j \boldsymbol{\sigma} \cdot \mathbf{g}^\alpha d\xi^3 = \boxed{\left( \tilde{n}^{\alpha\beta} + \lambda_\mu^\beta \tilde{m}^{\alpha\mu} \right)} \boldsymbol{\varphi}_{,\beta}$$

coupling

$$\tilde{\mathbf{m}}^\alpha = \frac{1}{\bar{j}} \int_{h_{min}}^{h_{max}} j \xi^3 \boldsymbol{\sigma} \cdot \mathbf{g}^\alpha d\xi^3$$

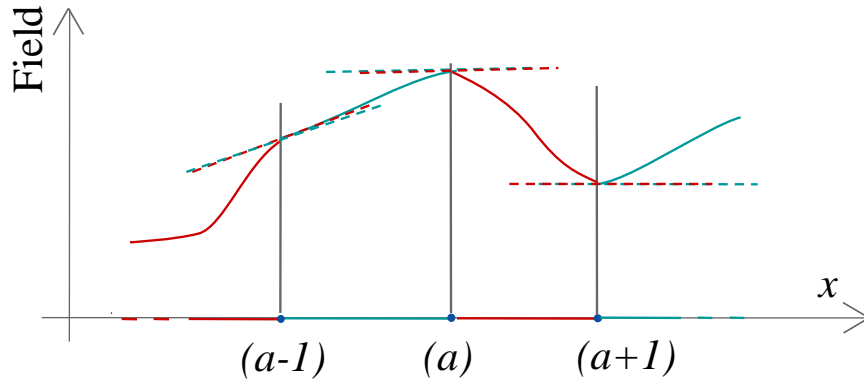
$$\mathbf{l} = \frac{1}{\bar{j}} \int_{h_{min}}^{h_{max}} j \boldsymbol{\sigma} \cdot \mathbf{g}^3 d\xi^3 \approx 0$$



- The (Simplified) equations of the problem are formulated in terms of the reduced stresses
  - Strong form  $\frac{1}{j} (\bar{j} \mathbf{n}^\alpha)_{,\alpha} + \frac{1}{j} (\bar{j} \tilde{\mathbf{m}}^\alpha) - \mathbf{l} = 0$
  - Weak form  $\int_A \left[ (\bar{j} \mathbf{n}^\alpha)_{,\alpha} \cdot \delta \boldsymbol{\varphi} + (\bar{j} \tilde{\mathbf{m}}^\alpha)_{,\alpha} \cdot \lambda_h \delta \mathbf{t} - \bar{j} \mathbf{l} \cdot \lambda_h \delta \mathbf{t} \right] dA = 0$
  - Highlights of the full DG concept
  - External forces and inertial terms are omitted (same as FEM)

# Full-DG formulation of Kirchhoff-Love shells

- FEM (Continuous Galerkin)

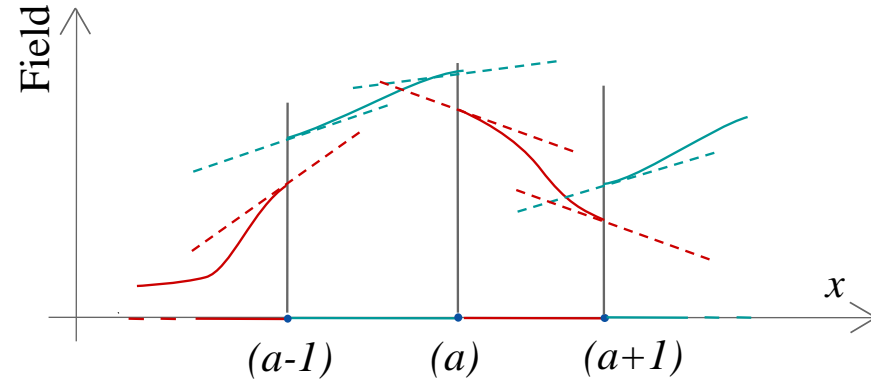


Integration by parts on the structure

$$\sum_e \int_{A_e} [\bar{j}n^\alpha \cdot \delta\varphi_{,\alpha} + \bar{j}\tilde{m}^\alpha \cdot \lambda_h \delta t_{,\alpha} - \bar{j}l \cdot \lambda_h \delta t] dA = 0$$

Additional interface terms exactly as for beams

- Discontinuous Galerkin



Integration by parts on each element (unusual on  $l$ )

$$\sum_e \left\{ \int_{A_e} \left[ (\bar{j}n^\alpha)_{,\alpha} \cdot \delta\varphi + (\bar{j}\tilde{m}^\alpha)_{,\alpha} \cdot \lambda_h \delta t - (\bar{j}l)_{,\alpha} \cdot \int_\alpha \lambda_h \delta t d\alpha' \right] dA \right.$$

$$\left. - \int_{\partial A_e} \left[ \bar{j}n^\alpha \cdot \delta\varphi v_\alpha^- + \bar{j}\tilde{m}^\alpha \cdot \lambda_h \delta t v_\alpha^- - \bar{j}l \cdot \int_\alpha \lambda_h \delta t d\alpha' v_\alpha^- \right] dA \right\} = 0$$

- The 3 interface terms are replaced by consistent numerical fluxes
  - Average fluxes are considered (exactly as for beams)

**Interface terms = A sum of jumps → Consistent terms**

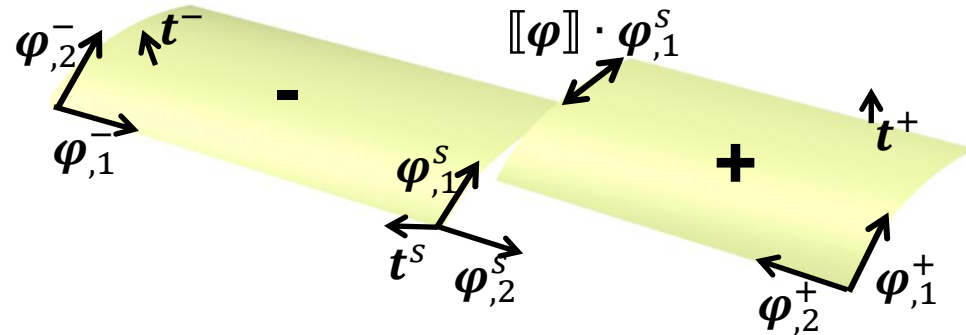
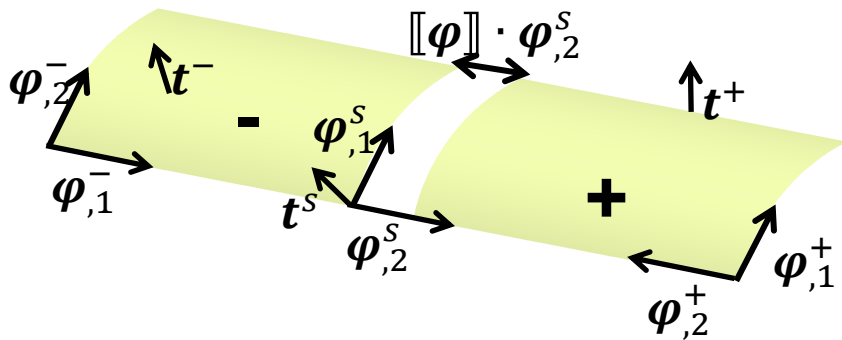
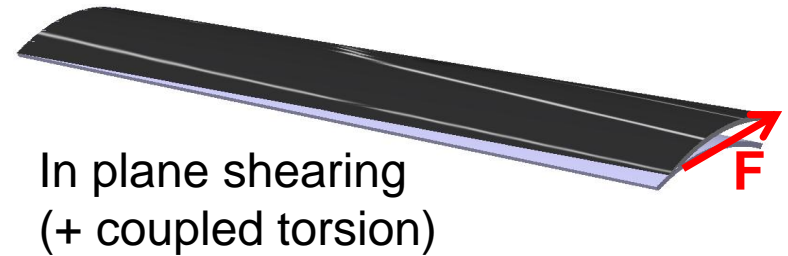
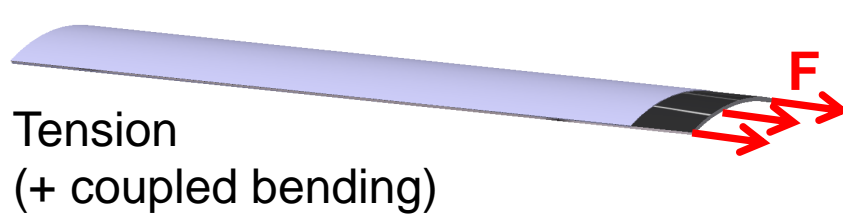
$$-\sum_e \int_{\partial A_e} \bar{j} n^\alpha \cdot \delta \varphi v_\alpha^- dA = \sum_s \int_s \llbracket \bar{j} n^\alpha \cdot \delta \varphi v_\alpha^- \rrbracket_s d\delta A_e \rightarrow \sum_s \int_s \langle \bar{j} n^\alpha \rangle \cdot \llbracket \delta \varphi \rrbracket v_\alpha^- d\partial A_e$$

$$-\sum_e \int_{\partial A_e} \bar{j} \tilde{m}^\alpha \cdot \lambda_h \delta t v_\alpha^- dA = \sum_s \int_s \llbracket \bar{j} \tilde{m}^\alpha \cdot \lambda_h \delta t v_\alpha^- \rrbracket_s d\delta A_e \rightarrow \sum_s \int_s \langle \bar{j} \tilde{m}^\alpha \rangle \cdot \llbracket \lambda_h \delta t \rrbracket v_\alpha^- d\partial A_e$$

$$\begin{aligned} \sum_e \int_{\partial A_e} \bar{j} l \cdot \int_\alpha \lambda_h \delta t d\alpha' v_\alpha^- dA &= - \sum_s \int_s \llbracket \bar{j} l \cdot \int_\alpha \lambda_h \delta t d\alpha' v_\alpha^- \rrbracket_s d\partial A_e \\ &\rightarrow - \sum_s \int_s \langle \bar{j} l \rangle \cdot \llbracket \int_\alpha \lambda_h \delta t d\alpha' \rrbracket v_\alpha^- d\partial A_e \approx 0 \end{aligned}$$

# Full-DG formulation of Kirchhoff-Love shells

- 3 Symmetrization terms are introduced to ensure (weakly) the continuity
  - The in-plane displacement jump is constrained by symmetrizing the consistency terms on  $\mathbf{n}^\alpha$



Consistency term  $\langle a \rangle \cdot [[\delta b]] \rightarrow$

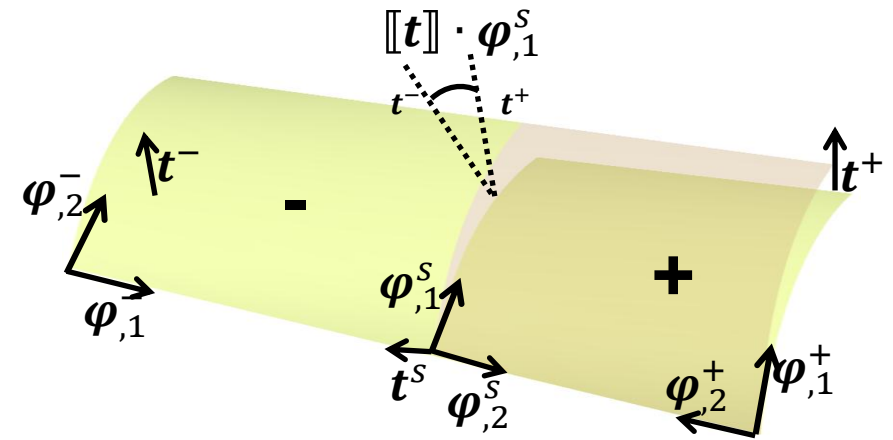
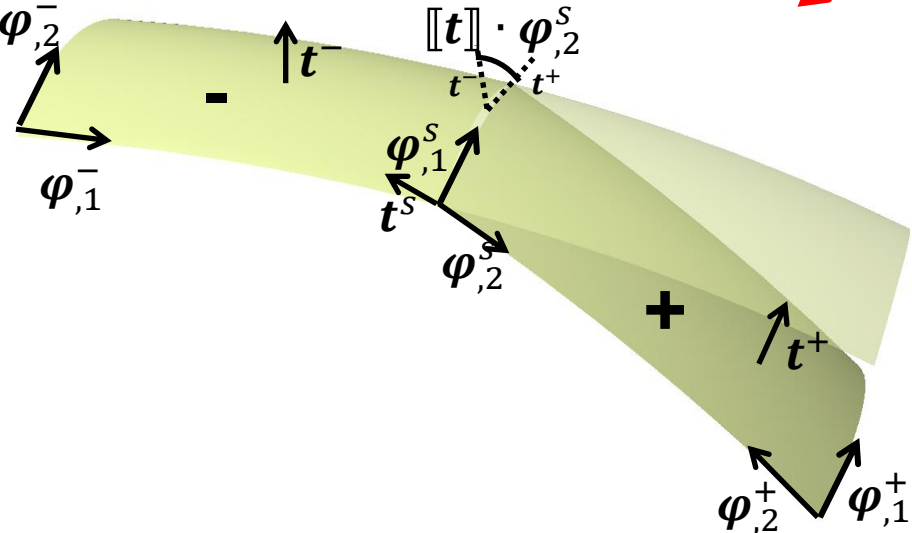
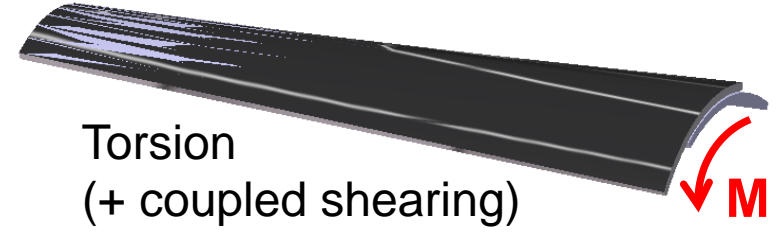
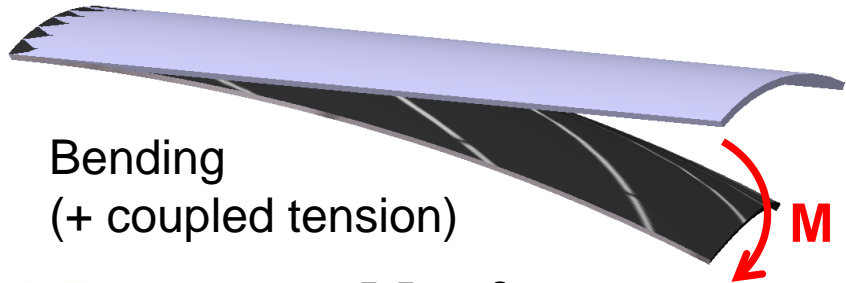
Symmetrization term  $\langle b \rangle \cdot [[\delta a]]$

$$\sum_s \int_s [[\boldsymbol{\varphi}]] \cdot \langle \delta(\bar{j} \mathbf{n}^\alpha) \rangle v_\alpha^- d\partial A_e = 0$$

- Leads to a symmetric formulation only in the linear case

# Full-DG formulation of Kirchhoff-Love shells

- 3 Symmetrization terms are introduced to ensure (weakly) the continuity
  - The rotational jump is constrained by symmetrizing the consistency terms on  $\tilde{m}^\alpha$

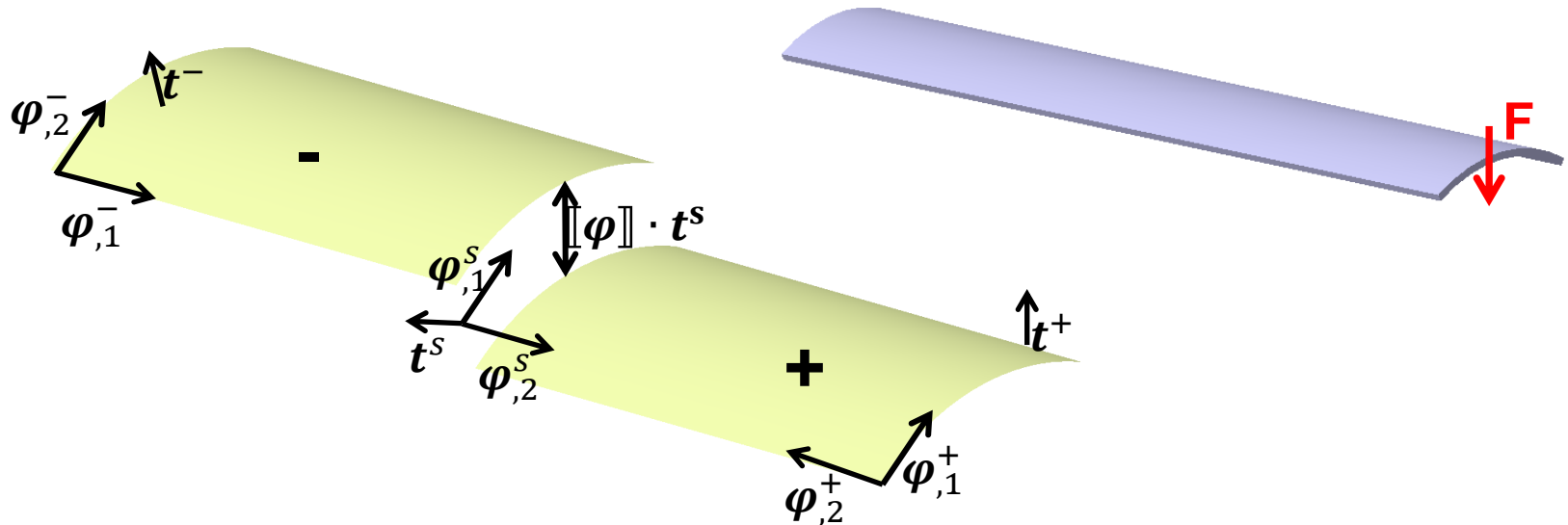


Consistency term  $\langle a \rangle \cdot [[\delta b]] \rightarrow$  Symmetrization term  $\langle b \rangle \cdot [[\delta a]]$

$$\sum_s \int_s [[\mathbf{t}]] \cdot \langle (j \lambda_h \tilde{m}^\alpha) \rangle \nu_\alpha^- d\partial A_e = 0$$

# Full-DG formulation of Kirchhoff-Love shells

- 3 Symmetrization terms are introduced to ensure (weakly) the continuity
  - The out-of-plane displacement jump is constrained by symmetrizing the consistency terms on  $l$



**Consistency term**  $\langle a \rangle \cdot [[\delta b]] \rightarrow$  **Symmetrization term**  $\langle b \rangle \cdot [[\delta a]]$

$$\sum_s \int_s \left[ \int_\alpha \lambda_h t d\alpha' \right] \cdot \langle \delta(jl) \rangle v_\alpha^- d\partial A_e = 0$$

$\lambda_h [[\varphi]] \cdot t \varphi^\alpha$  Primitive approximation

- 3 Stabilization terms have to be introduced to ensure the stability of the method
  - Quadratic terms are formulated from consistent and symmetrization terms in  $\mathbf{n}^\alpha$

**Form of stabilization terms**  $[[\mathbf{a}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta}{h^s} \text{Invariant stiff} \right\rangle [[\delta \mathbf{a}]] \cdot \boldsymbol{\varphi}_{,\beta} \nu_\alpha^-$

$$\left. \begin{aligned} & \sum_s \int_s \langle \bar{j} \mathbf{n}^\alpha \rangle \cdot [[\delta \boldsymbol{\varphi}]] \nu_\alpha^- d\partial A_e \\ & \sum_s \int_s [[\boldsymbol{\varphi}]] \cdot \langle \delta(\bar{j} \mathbf{n}^\alpha) \rangle \nu_\alpha^- d\partial A_e \end{aligned} \right\} \rightarrow \sum_s \left[ \int_s [[\boldsymbol{\varphi}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_2 \mathcal{H}_n^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle [[\delta \boldsymbol{\varphi}]] \cdot \boldsymbol{\varphi}_{,\beta} \nu_\alpha^- d\partial A_e = 0 \right]$$

- 3 Stabilization terms have to be introduced to ensure the stability of the method
  - Quadratic terms are formulated from consistent and symmetrization terms in  $\tilde{\mathbf{m}}^\alpha$

**Form of stabilization terms**  $[[\mathbf{a}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta}{h^s} \text{Invariant stiff} \right\rangle [[\delta \mathbf{a}]] \cdot \boldsymbol{\varphi}_{,\beta} \nu_\alpha^-$

$$\left. \begin{aligned} & \sum_s \int_s \langle \bar{j} \tilde{\mathbf{m}}^\alpha \rangle \cdot [[\lambda_h \delta \mathbf{t}]] \nu_\alpha^- d\partial A_e \\ & \sum_s \int_s [[\mathbf{t}]] \cdot \langle (\bar{j} \lambda_h \tilde{\mathbf{m}}^\alpha) \rangle \nu_\alpha^- d\partial A_e \end{aligned} \right\} \rightarrow \sum_s \int_s [[\mathbf{t}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_1 \mathcal{H}_m^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle [[\delta \mathbf{t}]] \cdot \boldsymbol{\varphi}_{,\beta} \nu_\alpha^- d\partial A_e = 0$$

- 3 Stabilization terms have to be introduced to ensure the stability of the method
  - Quadratic terms are formulated from consistent and symmetrization terms in  $\mathbf{l}$

**Form of stabilization terms**  $[[\mathbf{a}]] \cdot \mathbf{t} v_{\beta}^{-} \left\langle \frac{\beta}{h^s} \text{Invariant stiff} \right\rangle [[\delta \mathbf{a}]] \cdot \mathbf{t} v_{\alpha}^{-}$

$$\left. \begin{aligned} & \sum_s \int_s \langle \bar{j} \mathbf{l} \rangle \cdot \left[ \int_{\alpha} \lambda_h \delta \mathbf{t} d\alpha' \right] v_{\alpha}^{-} d\partial A_e \\ & \sum_s \int_s \lambda_h [[\boldsymbol{\varphi}]] \cdot \mathbf{t} \boldsymbol{\varphi}^{\alpha} \langle \delta(\bar{j} \mathbf{l}) \rangle v_{\alpha}^{-} d\partial A_e \end{aligned} \right\} \rightarrow \sum_s \int_s [[\boldsymbol{\varphi}]] \cdot \mathbf{t} v_{\beta}^{-} \left\langle \frac{\beta_3 \mathcal{H}_s^{\alpha\beta} \bar{j}_0}{h^s} \right\rangle [[\delta \boldsymbol{\varphi}]] \cdot \mathbf{t} v_{\alpha}^{-} d\partial A_e = 0$$

# Full-DG formulation of Kirchhoff-Love shells

- The terms of stabilization in  $l$  ensure also weakly the out-of-plane continuity

- The shearing is neglected (Kirchhoff-Love assumption)  $\rightarrow l \approx 0$

- Consistency terms

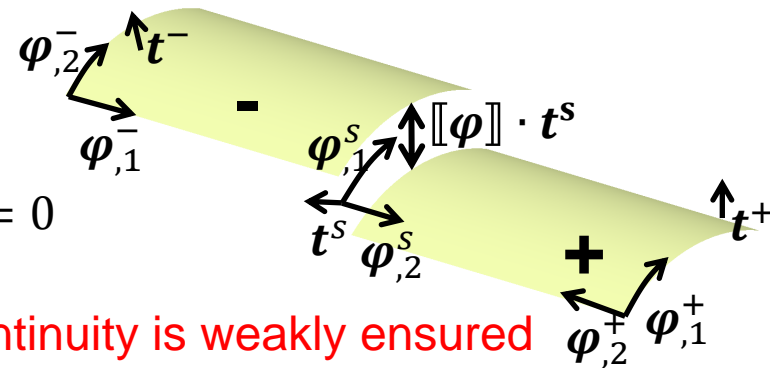
$$\int_s \langle \bar{j}l \rangle \cdot \left[ \int_\alpha \lambda_h \delta t d\alpha' \right] v_\alpha^- d\partial A_e \approx 0$$

- Symmetrization terms (if considered  $\rightarrow$  unsymmetrical formulation)

$$\int_s \left[ \int_\alpha \lambda_h t d\alpha' \right] \cdot \langle \delta(jl) \rangle v_\alpha^- d\partial A_e \approx 0$$

- Stabilization terms

$$\int_s \underbrace{[[\varphi]] \cdot t}_{\text{red circle}} v_\beta^- \left\langle \frac{\beta_3 \mathcal{H}_s^{\alpha\beta} \bar{j}_0}{h^s} \right\rangle [[\delta\varphi]] \cdot t v_\alpha^- d\partial A_e = 0$$



Out-of-plane displacement jump is constrained  $\rightarrow$  continuity is weakly ensured

# Full-DG formulation of Kirchhoff-Love shells

- The equation of the full-DG formulation is obtained by adding the different contributions [*Becker et al cmame2011, Becker et al ijmme2012*]

$$\sum_e \int_{A_e} [(\bar{j}\mathbf{n}^\alpha)_{,\alpha} \cdot \delta\boldsymbol{\varphi} + (\bar{j}\tilde{\mathbf{m}}^\alpha)_{,\alpha} \cdot \lambda_h \delta\mathbf{t}] dA + \text{FEM (CG) equation}$$

$$\begin{aligned} & \sum_s \int_s \left[ \langle \bar{j}\mathbf{n}^\alpha \rangle \cdot \llbracket \delta\boldsymbol{\varphi} \rrbracket + \llbracket \boldsymbol{\varphi} \rrbracket \cdot \langle \delta(\bar{j}\mathbf{n}^\alpha) \rangle + \llbracket \boldsymbol{\varphi} \rrbracket \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_2 \mathcal{H}_n^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle \llbracket \delta\boldsymbol{\varphi} \rrbracket \cdot \boldsymbol{\varphi}_{,\beta} \right] \nu_\alpha^- d\partial A_e + \\ & \sum_s \int_s \left[ \langle \bar{j}\tilde{\mathbf{m}}^\alpha \rangle \cdot \llbracket \lambda_h \delta\mathbf{t} \rrbracket + \llbracket \mathbf{t} \rrbracket \cdot \langle (\bar{j}\lambda_h \tilde{\mathbf{m}}^\alpha) \rangle + \llbracket \mathbf{t} \rrbracket \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_1 \mathcal{H}_m^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle \llbracket \delta\mathbf{t} \rrbracket \cdot \boldsymbol{\varphi}_{,\beta} \right] \nu_\alpha^- d\partial A_e + \\ & \sum_s \int_s \llbracket \boldsymbol{\varphi} \rrbracket \cdot \mathbf{t} \nu_\beta^- \left\langle \frac{\beta_3 \mathcal{H}_s^{\alpha\beta} \bar{j}_0}{h^s} \right\rangle \llbracket \delta\boldsymbol{\varphi} \rrbracket \cdot \mathbf{t} \nu_\alpha^- d\partial A_e = 0 \end{aligned}$$

Consistency terms
Symmetrization terms
Stabilization terms

- Similar form as the beam case (2 Bulk, 2 consistency, 2 symmetrization and 3 stabilization terms)

# Full-DG formulation of Kirchhoff-Love shells

- The  $C^0$ /DG formulation [Noels et al cmame2008, Noels ijmme2009] is found if continuous elements are used ( $[[\boldsymbol{\varphi}]] = [[\boldsymbol{\delta}\boldsymbol{\varphi}]] = 0$ )

$$\begin{aligned}
 & \sum_e \int_{A_e} [(\bar{j}\mathbf{n}^\alpha)_{,\alpha} \cdot \boldsymbol{\delta}\boldsymbol{\varphi} + (\bar{j}\tilde{\mathbf{m}}^\alpha)_{,\alpha} \cdot \lambda_h \boldsymbol{\delta}\mathbf{t}] dA + \\
 & \sum_s \int_s \left[ \langle \bar{j}\mathbf{n}^\alpha \rangle \cdot [[\boldsymbol{\delta}\boldsymbol{\varphi}]] + [[\boldsymbol{\varphi}]] \cdot \langle \boldsymbol{\delta}(\bar{j}\mathbf{n}^\alpha) \rangle + \langle \bar{j}\tilde{\mathbf{m}}^\alpha \rangle \cdot [[\lambda_h \boldsymbol{\delta}\mathbf{t}]] + [[\mathbf{t}]] \cdot \langle (\bar{j}\lambda_h \tilde{\mathbf{m}}^\alpha) \rangle \right. \\
 & \left. + [[\boldsymbol{\varphi}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_2 \mathcal{H}_n^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle [[\boldsymbol{\delta}\boldsymbol{\varphi}]] \cdot \boldsymbol{\varphi}_{,\beta} \right] \nu_\alpha^- d\partial A_e + \\
 & \left. + \left[ \langle \bar{j}\tilde{\mathbf{m}}^\alpha \rangle \cdot [[\lambda_h \boldsymbol{\delta}\mathbf{t}]] + [[\mathbf{t}]] \cdot \langle (\bar{j}\lambda_h \tilde{\mathbf{m}}^\alpha) \rangle + [[\mathbf{t}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_1 \mathcal{H}_m^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle [[\boldsymbol{\delta}\mathbf{t}]] \cdot \boldsymbol{\varphi}_{,\beta} \right] \nu_\alpha^- d\partial A_e + \right. \\
 & \left. + \left[ \int_s [[\boldsymbol{\varphi}]] \cdot \mathbf{t} \nu_\beta^- \left\langle \frac{\beta_3 \mathcal{H}_s^{\alpha\beta} \bar{j}_0}{h^s} \right\rangle [[\boldsymbol{\delta}\boldsymbol{\varphi}]] \cdot \mathbf{t} \right] \nu_\alpha^- d\partial A_e = 0
 \end{aligned}$$

Consistency terms
Symmetrization terms
Stabilization terms

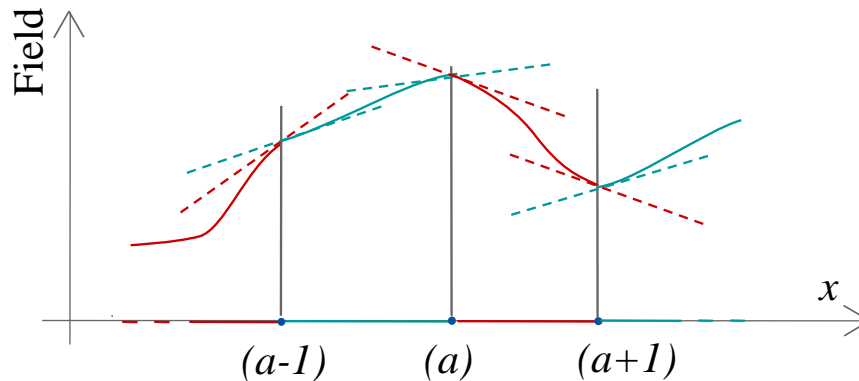
# Full-DG formulation of Kirchhoff-Love shells

- The  $C^0$ /DG formulation [Noels et al cmame2008, Noels ijmme2009] is found if continuous elements are used ( $[[\boldsymbol{\varphi}]] = [[\delta\boldsymbol{\varphi}]] = 0$ )

$$\sum_e \int_{A_e} [(\bar{j}\mathbf{n}^\alpha)_{,\alpha} \cdot \delta\boldsymbol{\varphi} + (\bar{j}\tilde{\mathbf{m}}^\alpha)_{,\alpha} \cdot \lambda_h \delta\mathbf{t}] dA + \sum_s \int_s \left[ \langle \bar{j}\tilde{\mathbf{m}}^\alpha \rangle \cdot [[\lambda_h \delta\mathbf{t}]] + [[\mathbf{t}]] \cdot \langle (\bar{j}\lambda_h \tilde{\mathbf{m}}^\alpha) \rangle + [[\mathbf{t}]] \cdot \boldsymbol{\varphi}_{,\gamma} \nu_\delta^- \left\langle \frac{\beta_1 \mathcal{H}_m^{\alpha\beta\gamma\delta} \bar{j}_0}{h^s} \right\rangle [[\delta\mathbf{t}]] \cdot \boldsymbol{\varphi}_{,\beta} \right] \nu_\alpha^- d\partial A_e = 0$$

Consistency terms
Symmetrization terms
Stabilization terms

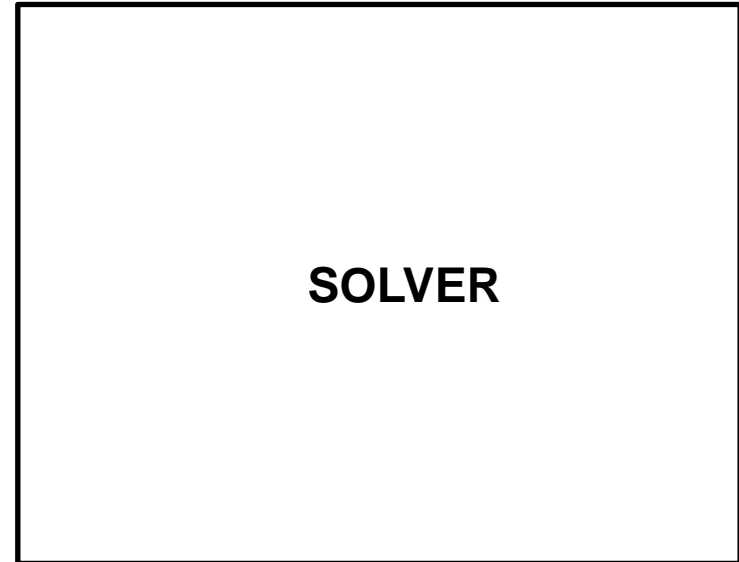
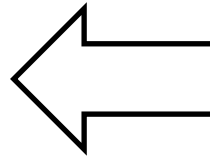
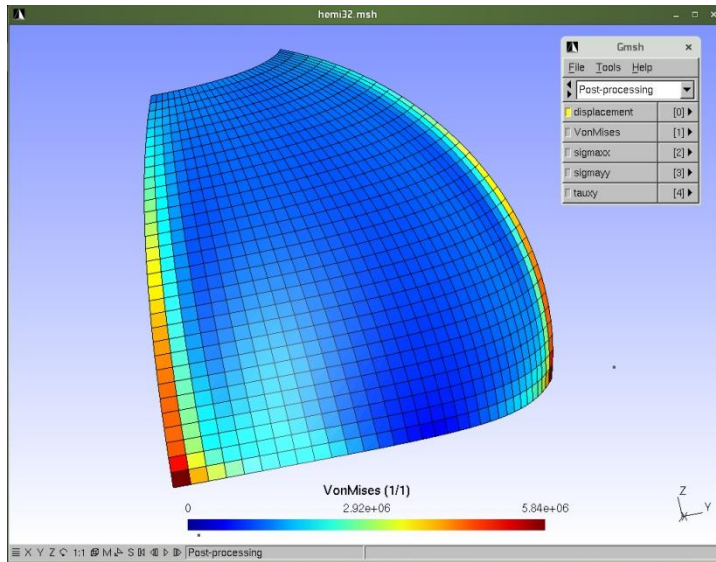
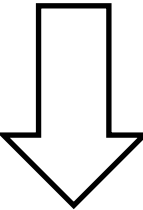
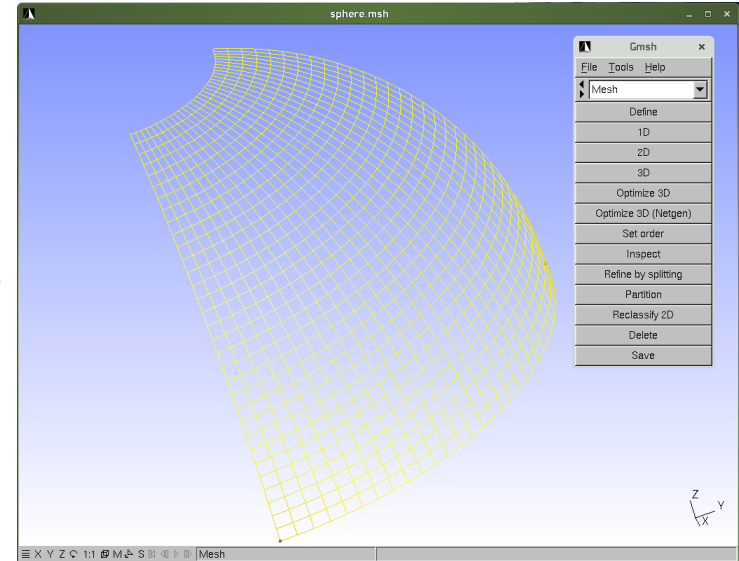
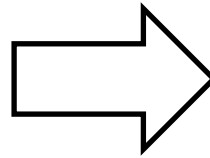
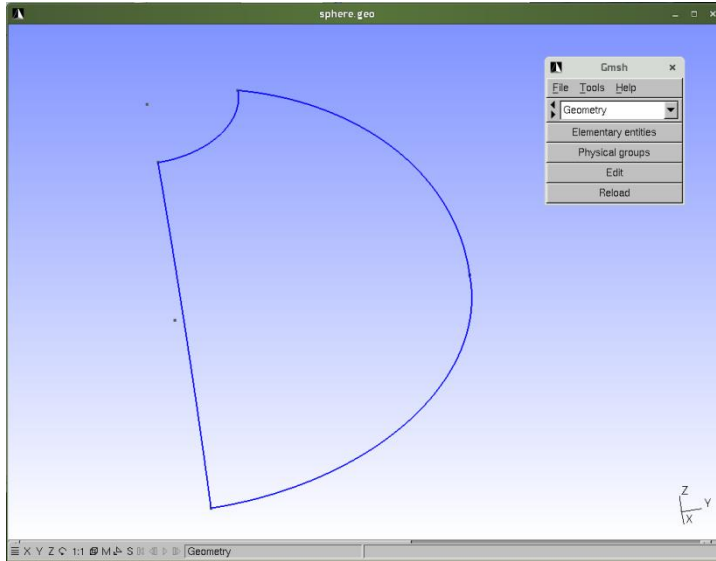
- Elements are continuous but the tangent continuity is ensured by DG



- The implementation is based on Gmsh
  - 3D finite element grid generator with a built-in CAD engine and a post-processor
  - Developed by C. Geuzaine (Ulg) and J.-F. Remacle (Ucl) [*Geuzaine et al ijnme2009*]
  - Industrially used (Cenaero, EDF, ...)

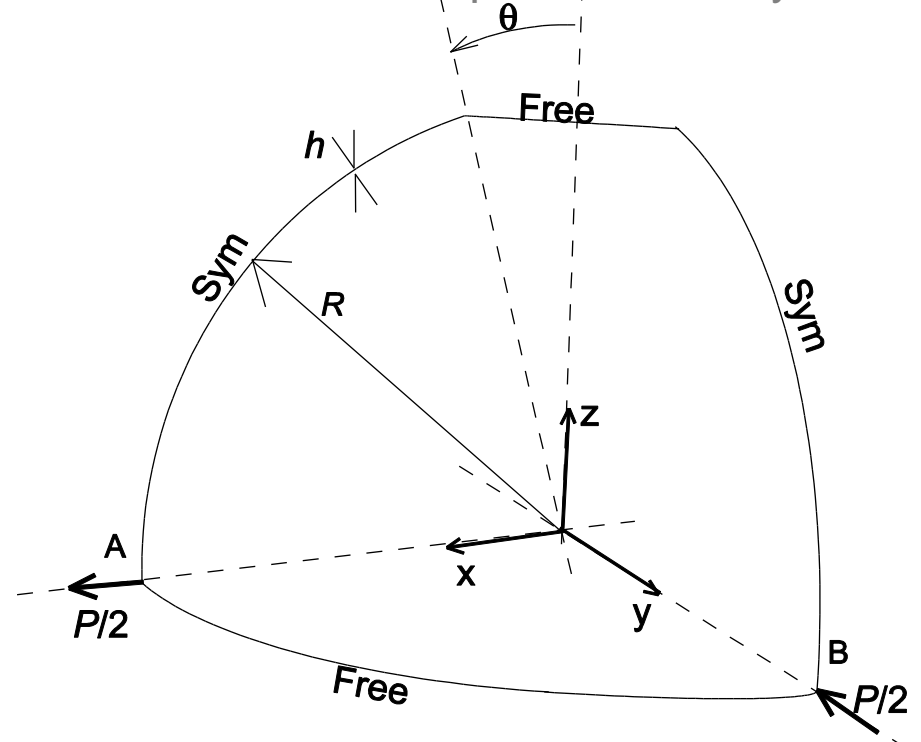
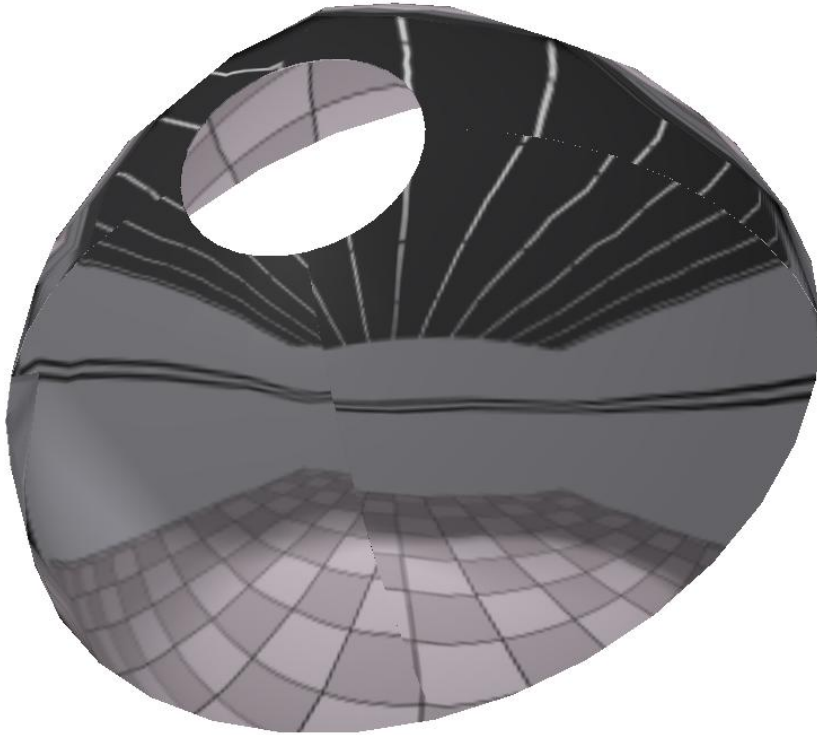
# Full-DG formulation of Kirchhoff-Love shells

- Elements & post-processing C++ classes of Gmsh are used in the solver



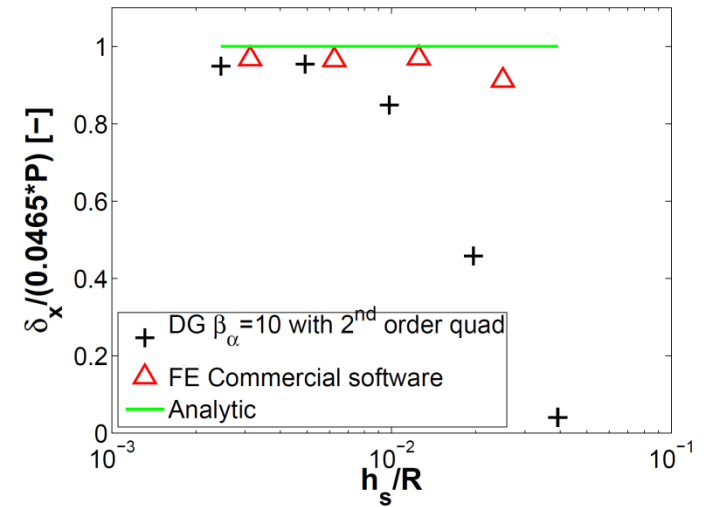
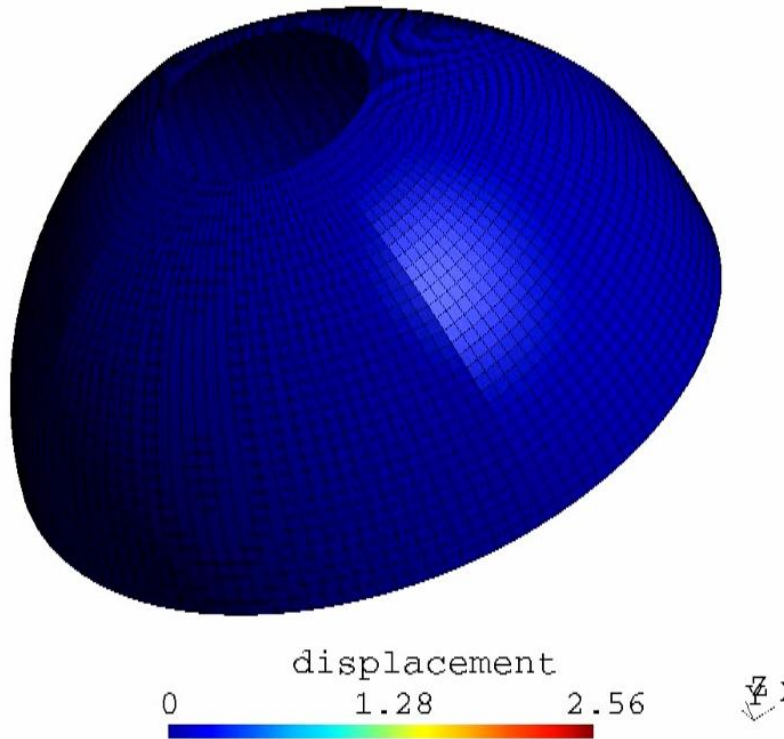
# Full-DG formulation of Kirchhoff-Love shells

- 2 benchmarks to prove the ability of the full-DG formulation to model continuous mechanics
  - Elastic open hemisphere with small strains loaded in a quasi-static way



# Full-DG formulation of Kirchhoff-Love shells

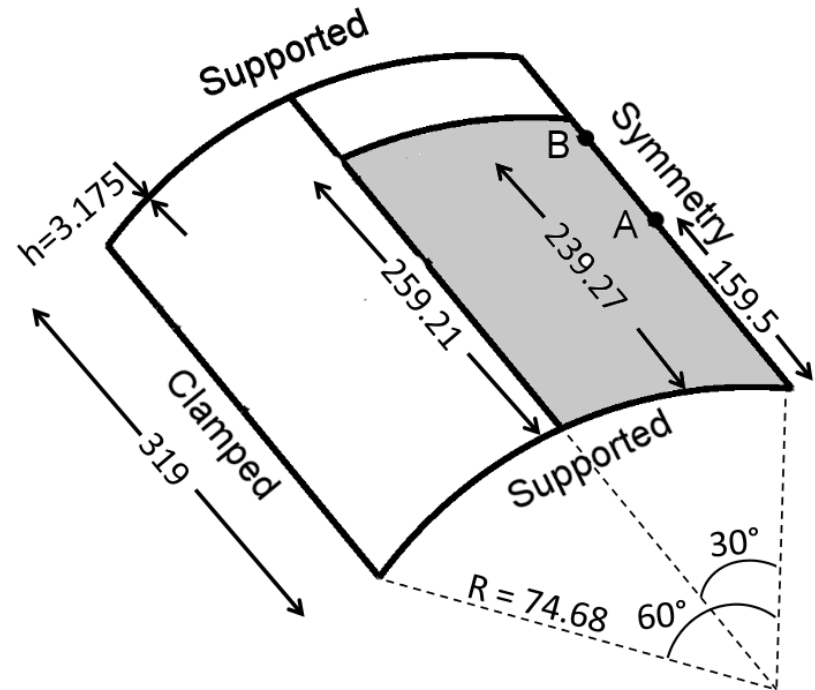
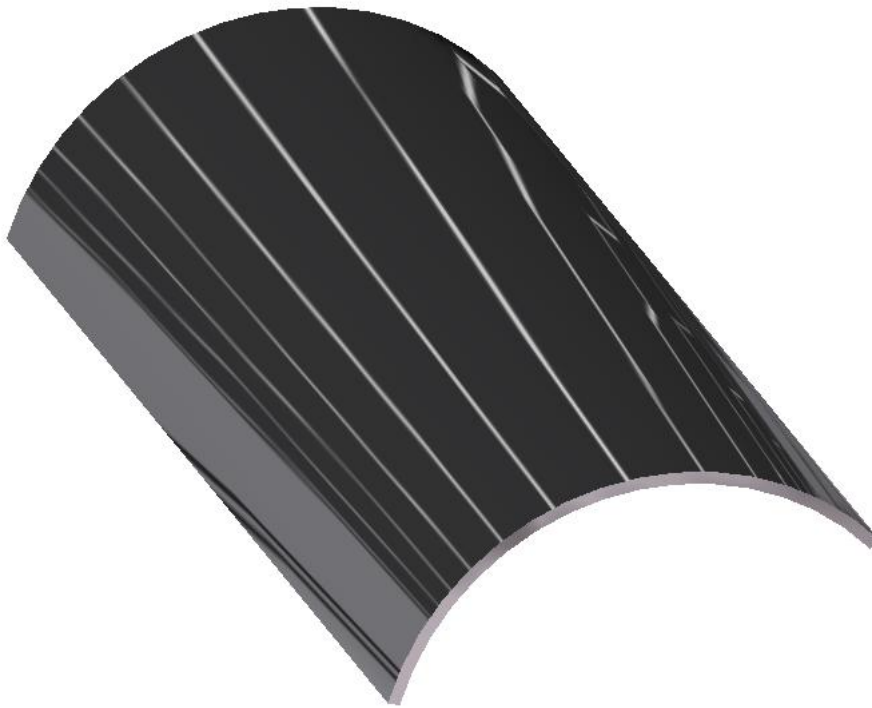
- 2 benchmarks to prove the ability of the full-DG formulation to model continuous mechanics
  - Elastic open hemisphere with small strains loaded in a quasi-static way



- The method converges to the analytical solution with the mesh refinement

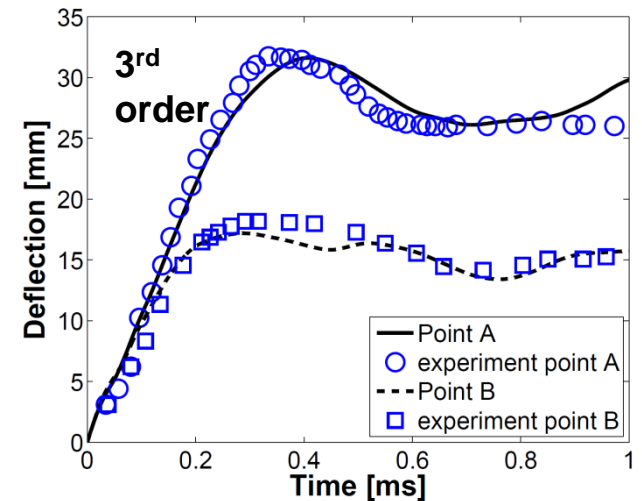
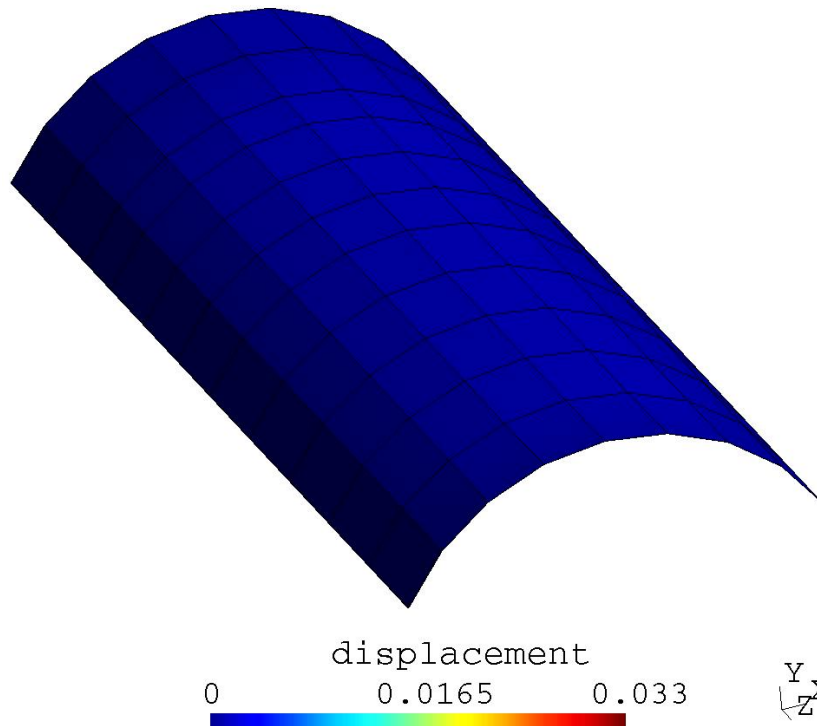
# Full-DG formulation of Kirchhoff-Love shells

- 2 benchmarks to prove the ability of the full-DG formulation to model continuous mechanics
  - $J_2$ -linear hardening (elasto-plastic large deformations) panel loaded dynamically (explicit Hulbert-Chung scheme)



# Full-DG formulation of Kirchhoff-Love shells

- 2 benchmarks to prove the ability of the full-DG formulation to model continuous mechanics
  - $J_2$ -linear hardening (elasto-plastic large deformations) panel loaded dynamically (explicit Hulbert-Chung scheme)



- The results match experimental data

## Full-DG formulation of Kirchhoff-Love shells

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- The full-DG method provides accurate results but is more costly than  $C^0$ /DG (memory, computational time) as it considers more degrees of freedom
  - Number of dofs (for the same mesh)

Benchmark	$C^0$ /DG	Full-DG
Open hemisphere	867	1728
Cylindrical panel	1683	3456

- The number of dofs is more or less twice larger for the full-DG formulation

- The full-DG method can be advantageously used for
  - Parallel computation for explicit scheme [*Becker et al, ijmme2012*]
  - Fracture applications (same number of Dofs as FEM/ICL)

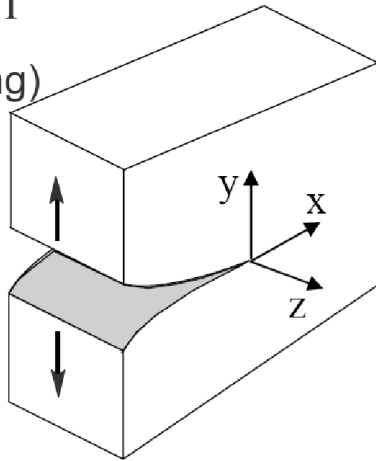
# Plan

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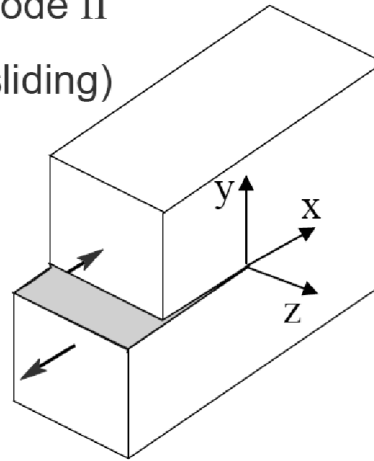
- Develop a discontinuous Galerkin method for thin bodies
  - Beam elements (1.5D case)
  - Shell elements (2.5D case)
- Discontinuous Galerkin / Extrinsic Cohesive law framework
  - Develop a suitable cohesive law for thin bodies
- Applications
  - Fragmentations, crack propagations under blast loadings

- There are 3 fracture modes in fracture mechanics

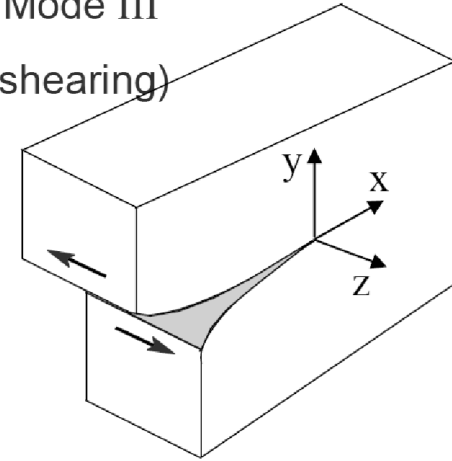
Mode I  
(opening)



Mode II  
(sliding)

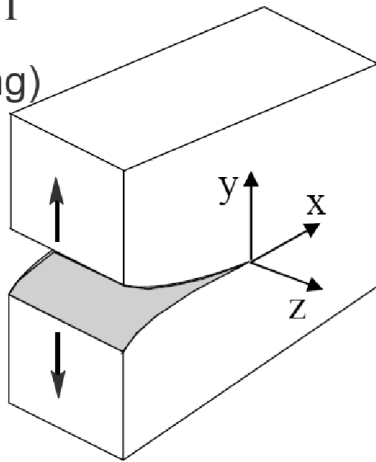


Mode III  
(shearing)

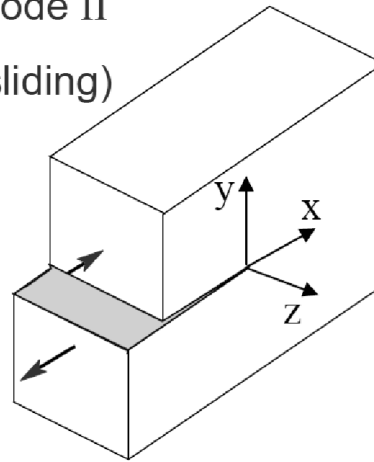


- Only modes I and II can be modeled by Kirchhoff-Love theory
  - Kirchhoff-Love  $\rightarrow$  out-of-plane shearing is neglected

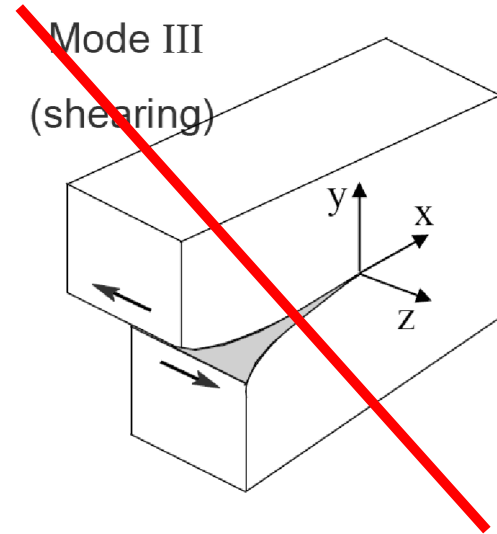
Mode I  
(opening)



Mode II  
(sliding)

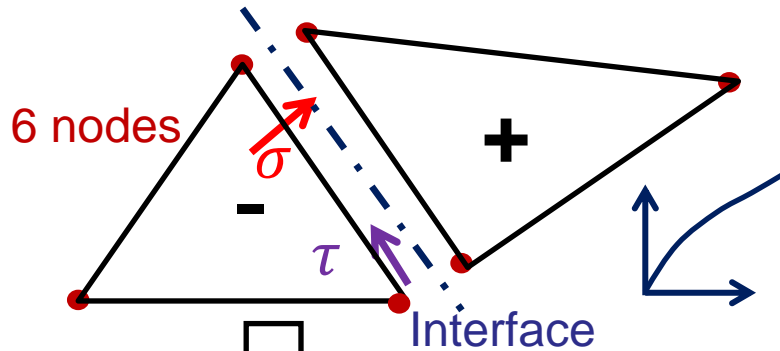


~~Mode III  
(shearing)~~

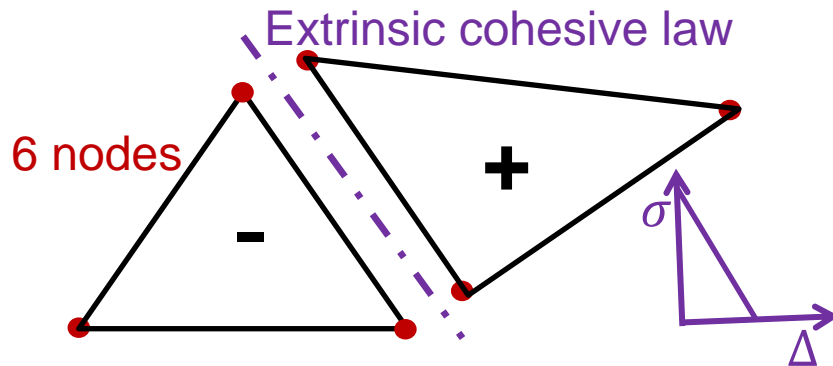


- Model restricted to problems with negligible 3D effects at the crack tip

- Fracture criterion based on an effective stress
  - Camacho & Ortiz Fracture criterion [*Camacho et al ijss1996*]

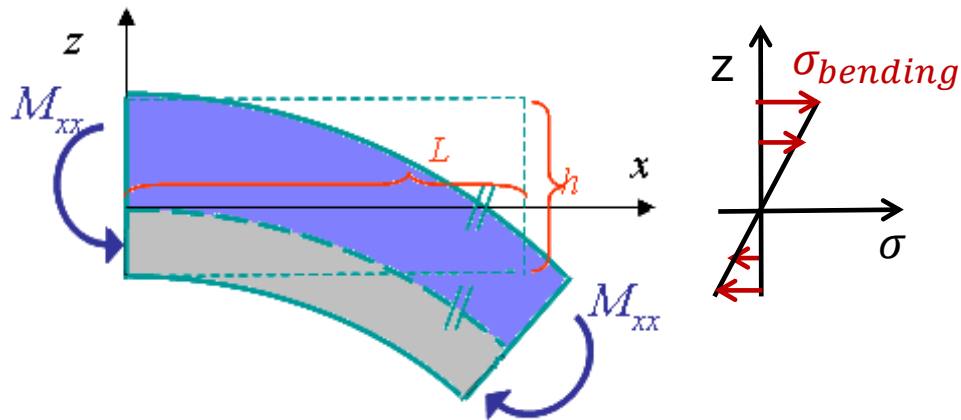


$$\sigma_{eff} > \sigma_c \quad \text{with} \quad \sigma_{eff} = \begin{cases} \sqrt{\sigma^2 + \beta^{-2}\tau^2} & \text{if } \sigma \geq 0 \quad \text{Traction} \\ \frac{1}{\beta} \ll |\tau| - \mu_c |\sigma| & \text{if } \sigma < 0 \quad \text{Compression} \end{cases}$$

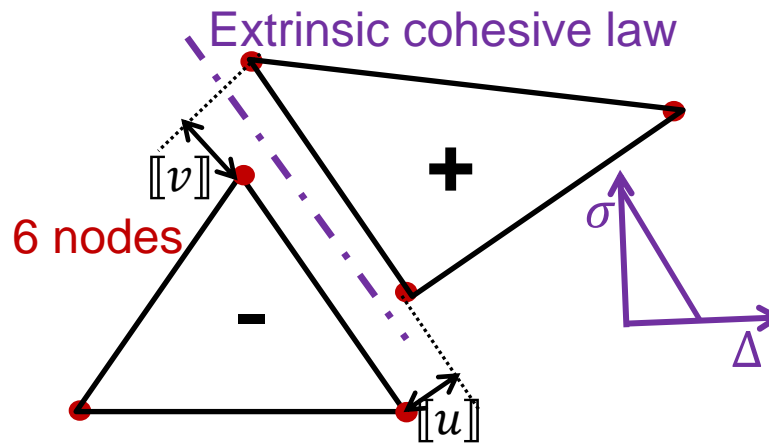


$\sigma_c, \beta$  and  $\mu_c$  are material parameters

- The effective stress is evaluated at the external fibers
  - The bending stress varies along the thickness
  - The fracture criterion is evaluated where the stress is maximum

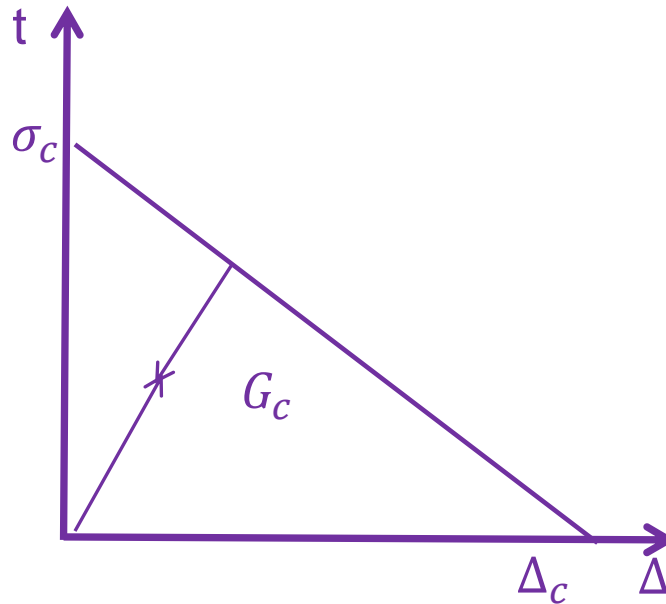


- The cohesive law is formulated in terms of an effective opening
  - Camacho & Ortiz Fracture criterion [*Camacho et al ijss1996*]

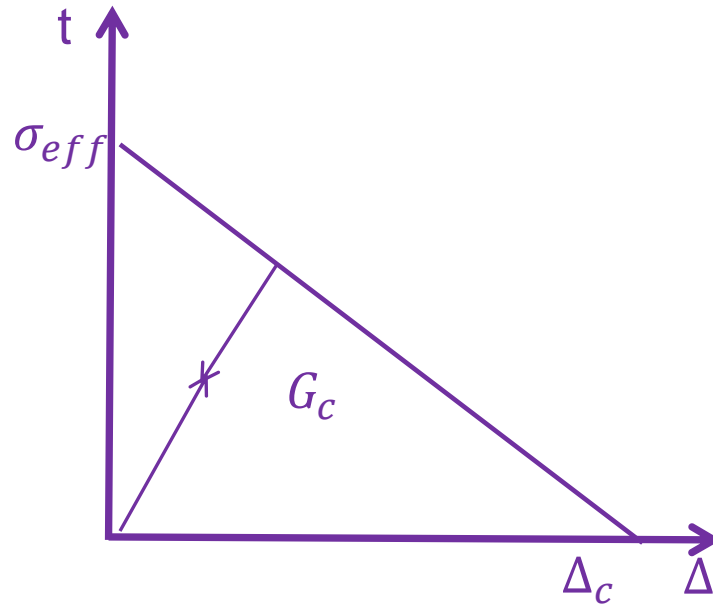


$$\Delta = \sqrt{[[u]] + \beta^2 [[v]]}$$

- The area under the cohesive law has to be equal to the fracture energy  $G_c$ 
  - $G_c$  is a material parameter

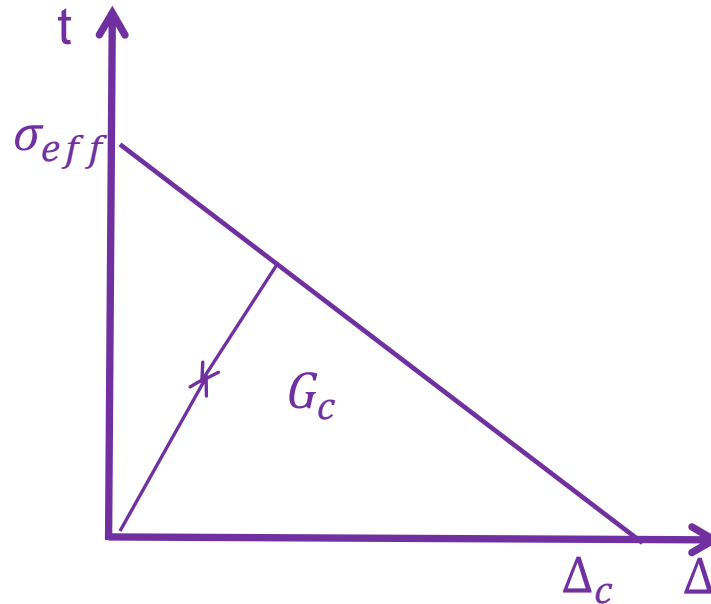


- The maximal stress of the cohesive law is equal to  $\sigma_{eff}$ 
  - Ensure the continuity of stresses



- Otherwise numerical problems [*Papoulia et al ijmme2003*]

- The shape of the cohesive law is linearly decreasing
  - Little influence of the shape for brittle materials



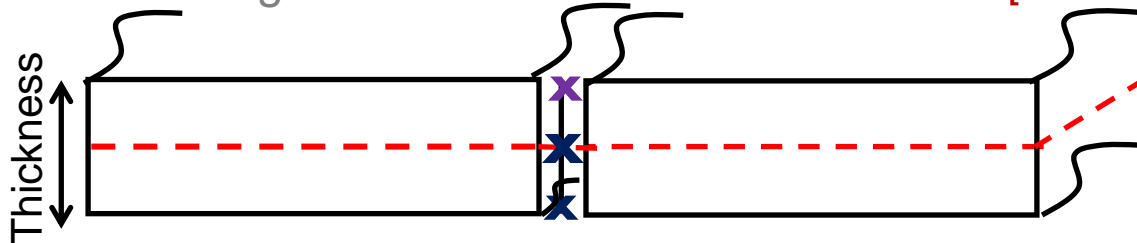
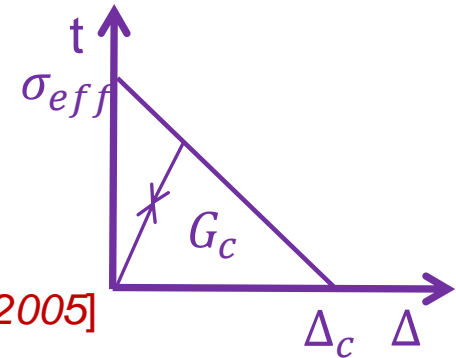
- $\Delta_c$  is equal to  $2G_c/\sigma_c$

# Full-DG/ECL framework

- The through the thickness crack propagation is not straightforward with shell elements

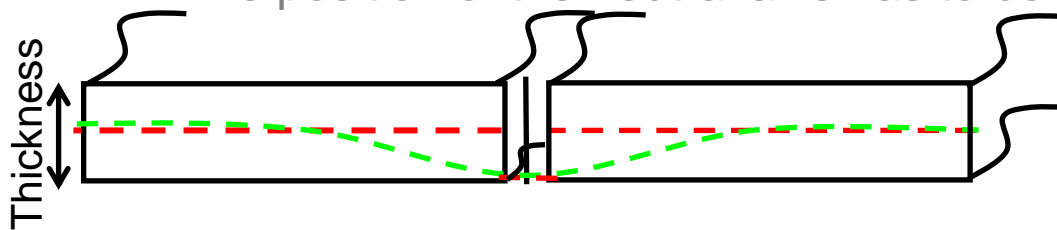
- No elements on thickness

- Integrate the 3D TSL on the thickness [Cirak et al cmame2005]



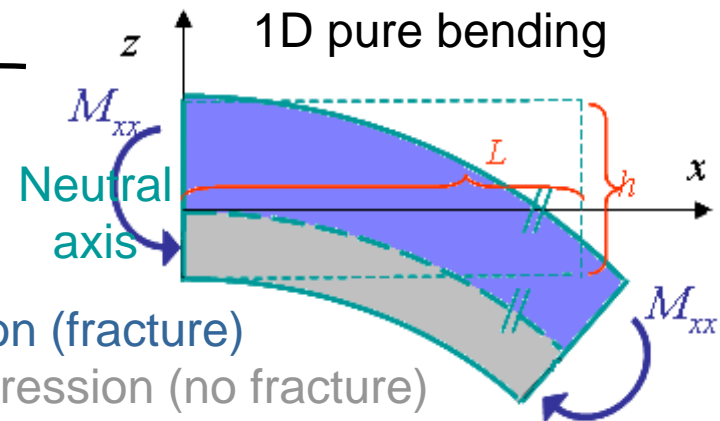
Fracture criterion is met  
 → cohesive law  
 Unreached fracture  
 → bulk law

- The position of the neutral axis has to be recomputed to propagate the crack



Discontinuity

Continuity (Computation ?)

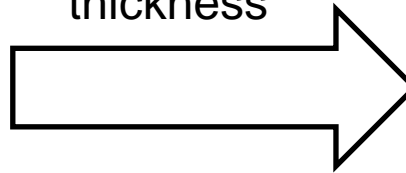


- The cohesive law can be formulated in terms of reduced stresses

- Same as shell equations

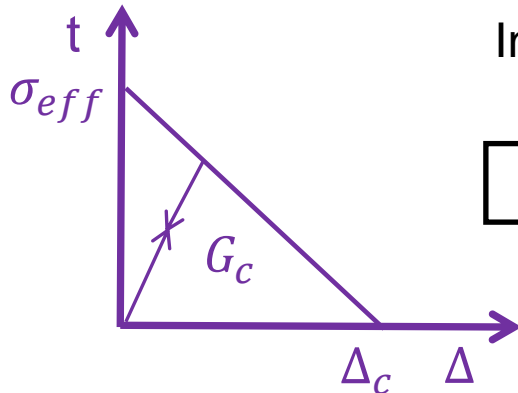
Bulk law  
Stress tensor  $\sigma$

Integration on  
thickness

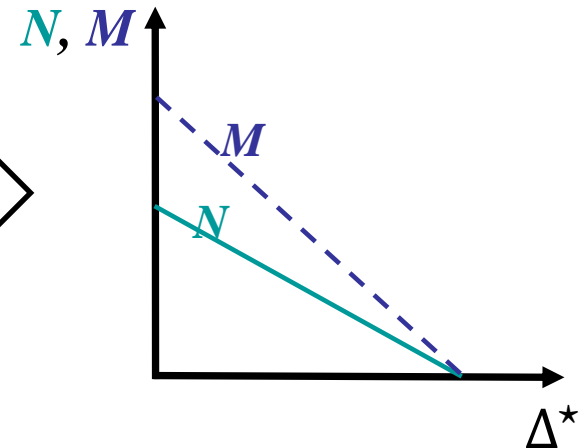
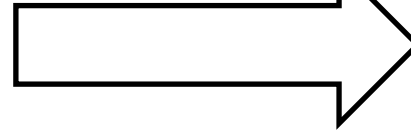


$$\mathbf{n}^\alpha = \frac{1}{j} \int_{h_{\min}}^{h_{\max}} j \boldsymbol{\sigma} \cdot \mathbf{g}^\alpha d\xi^3$$

$$\tilde{\mathbf{m}}^\alpha = \frac{1}{j} \int_{h_{\min}}^{h_{\max}} j \xi^3 \boldsymbol{\sigma} \cdot \mathbf{g}^\alpha d\xi^3$$



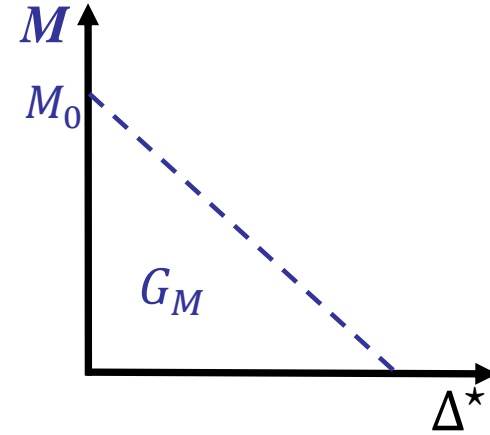
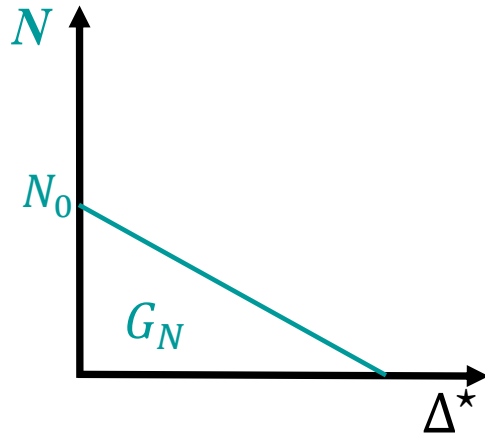
Integration on  
thickness



- Similar concept suggested by Zavattieri [[Zavattieri jam2006](#)]

- Define  $\Delta^*$  and  $N(\Delta^*)$ ,  $M(\Delta^*)$  to dissipate an energy equal to  $hG_c$  during the fracture process [*Becker et al ijmme2012*, *Becker et al ijf2012*]

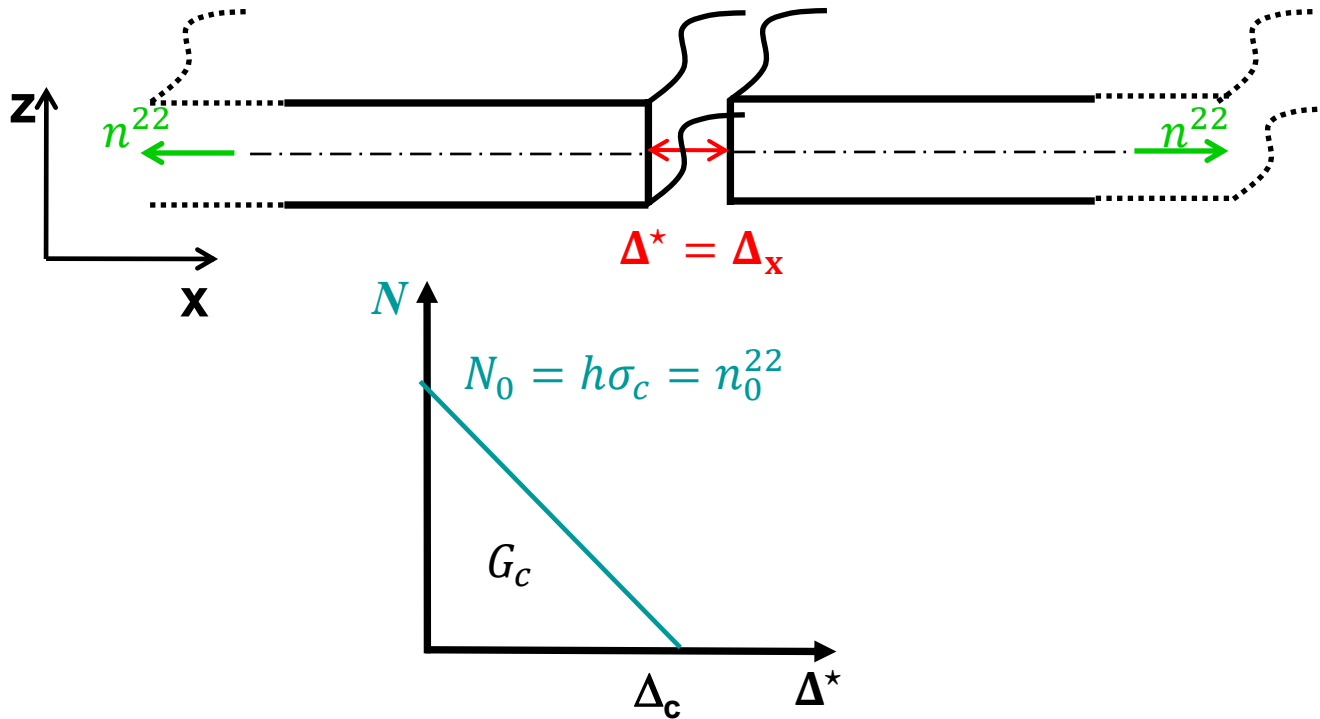
– Integration on thickness



$$G_N + G_M = hG_c$$

- The law  $N(\Delta^*)$  is defined to release an energy equal to  $hG_c$  in pure tension

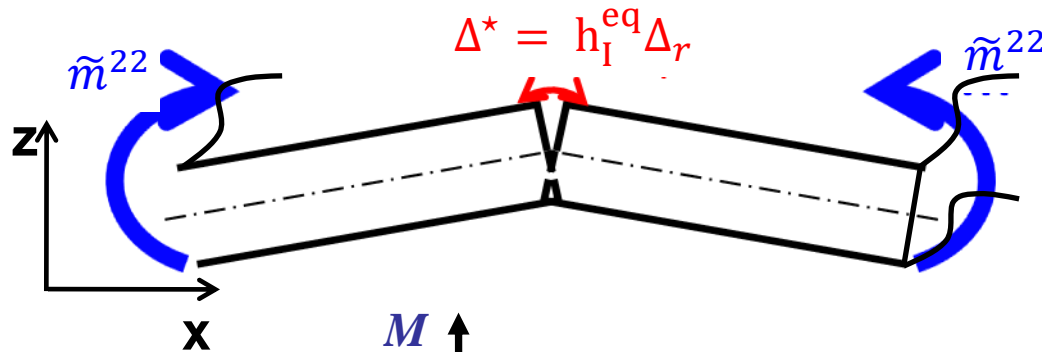
– Pure mode I



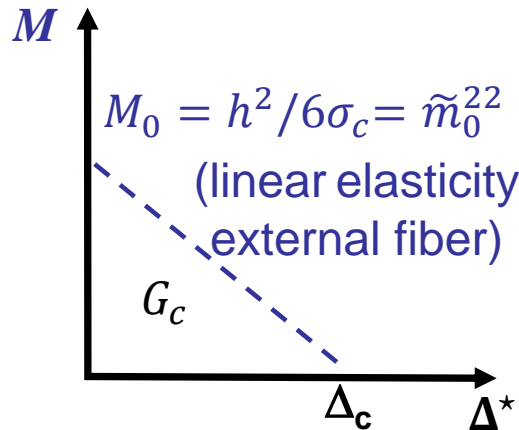
$$\int_0^{\Delta_c} N(\Delta_x) d\Delta_x = \frac{N_0 \Delta_c}{2} = \frac{2h\sigma_c G_c}{2\sigma_c} = hG_c$$

- The law  $M(\Delta^*)$  is defined to release an energy equal to  $hG_c$  in pure bending

– Pure mode I



$$\frac{\tilde{m}_0^{22}}{h_I^{eq}} = h\sigma_c$$

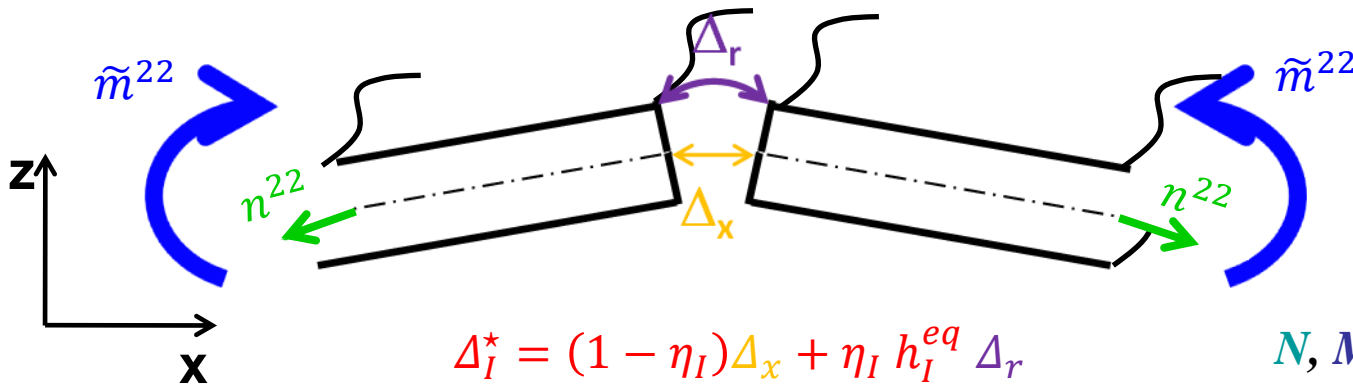


$$h_I^{eq} = \frac{h}{6}$$

$$\int_0^{\Delta_{rc}} M(\Delta^*) d\Delta_r = \int_0^{\Delta_c} \pm \frac{6}{h} M_0 \left(1 - \frac{\Delta^*}{\Delta_c}\right) d\Delta^* = \frac{6}{h} \frac{h^2 \sigma_c}{6} \frac{\Delta_c}{2} = hG_c$$

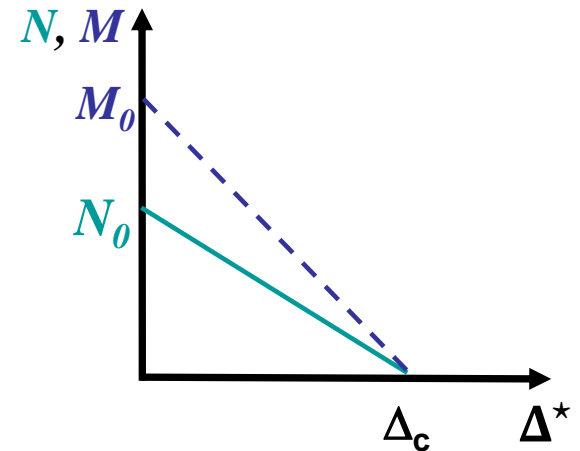
- Using the superposition principle the energy released for any couple  $N, M$  is equal to  $hG_c$  [Becker et al *ijnme2011*]

– Pure mode I



$$\Delta_I^* = (1 - \eta_I)\Delta_x + \eta_I h_I^{eq} \Delta_r$$

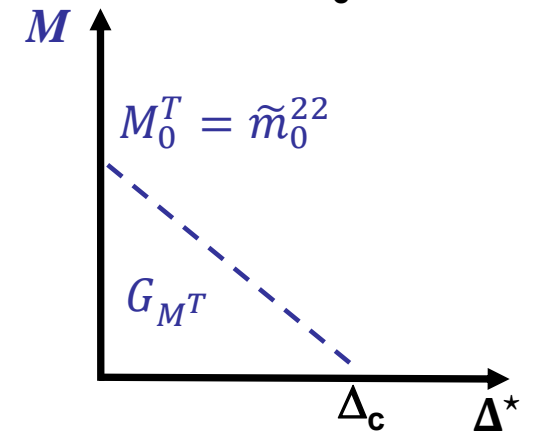
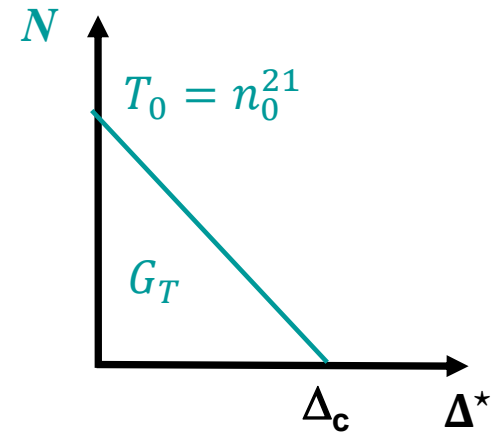
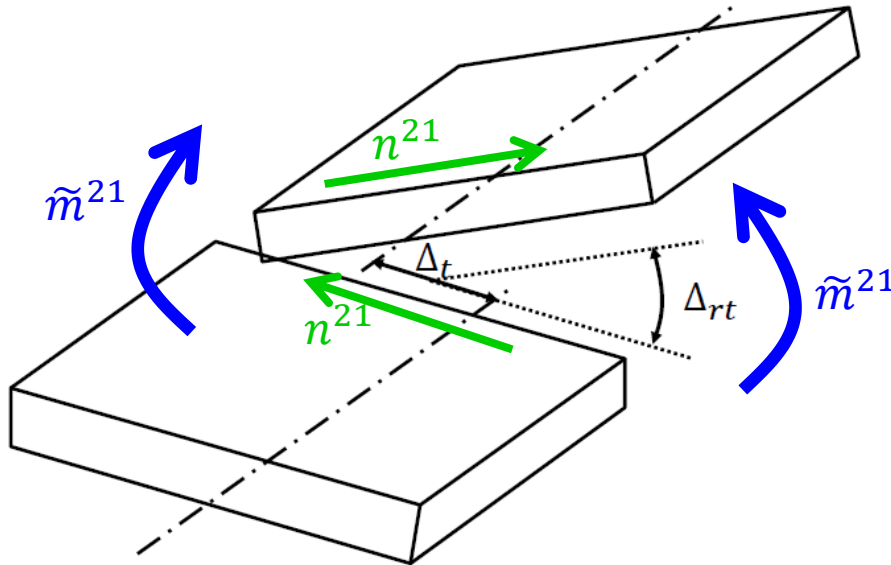
$$h_I^{eq} = \frac{M_0}{h\sigma_c - N_0}$$



– Coupling parameter

$$\eta_I = \frac{|1/h_I^{eq} M_0|}{N_0 + |1/h_I^{eq} M_0|} = \frac{h\sigma_c - N_0}{h\sigma_c}$$

- The cohesive model for mode I can be extended to mode II



$$G_T + G_{M^T} = h\beta G_c$$

$$\Delta_{II}^* = (1 - \eta_{II})\Delta_t + \eta_{II}h_{II}^{eq}\Delta_{rt}$$

$$h_{II}^{eq} = \frac{M_0^T}{h\beta\sigma_c - T_0}$$

– Coupling parameter

$$\eta_{II} = \frac{|1/h_{II}^{eq}M_0^T|}{T_0 + |1/h_{II}^{eq}M_0^T|} = \frac{h\beta\sigma_c - T_0}{h\beta\sigma_c}$$

- Combination of mode I and II is performed following Camacho & Ortiz [*Camacho et al ijss1996*]

- Usually perform in the literature

- Define an effective opening  $\Delta^* = \sqrt{\langle\langle \Delta_I^* \rangle\rangle^2 + \beta^2 \Delta_{II}^{*2}}$

- Fracture initiation  $\sigma_{eff} = \begin{cases} \sqrt{\sigma_I^2 + \beta^{-2}\tau_{II}^2} & \text{if } \sigma_I \geq 0 \\ \frac{1}{\beta} \langle\langle |\tau_{II}| - \mu_c |\sigma_I| \rangle\rangle & \text{if } \sigma_I < 0 \end{cases} = \sigma_c$

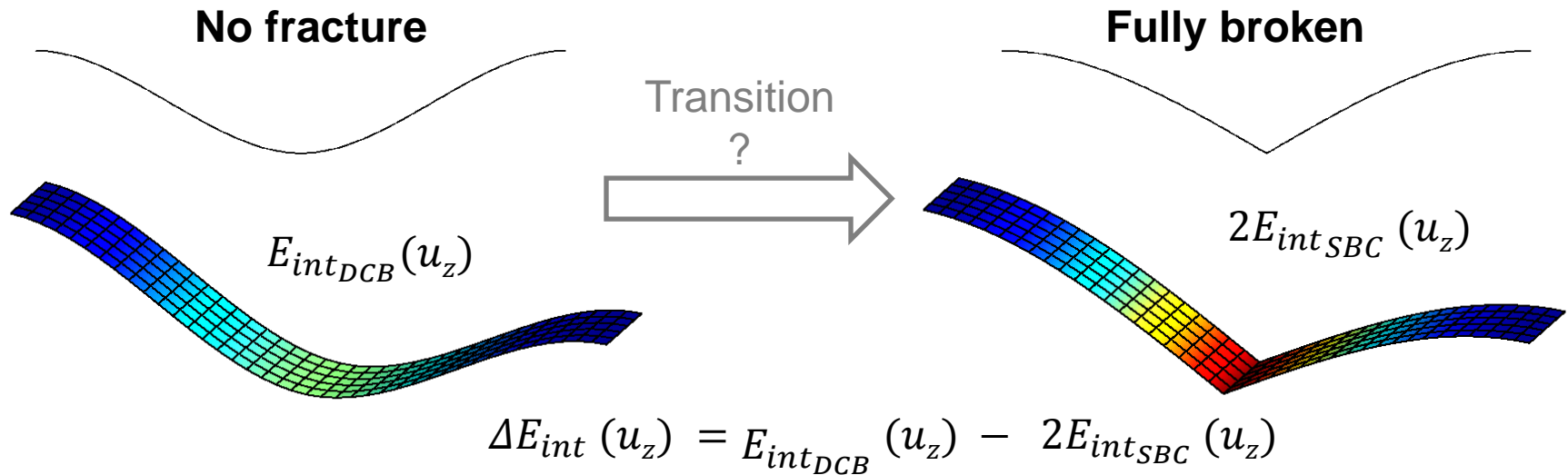
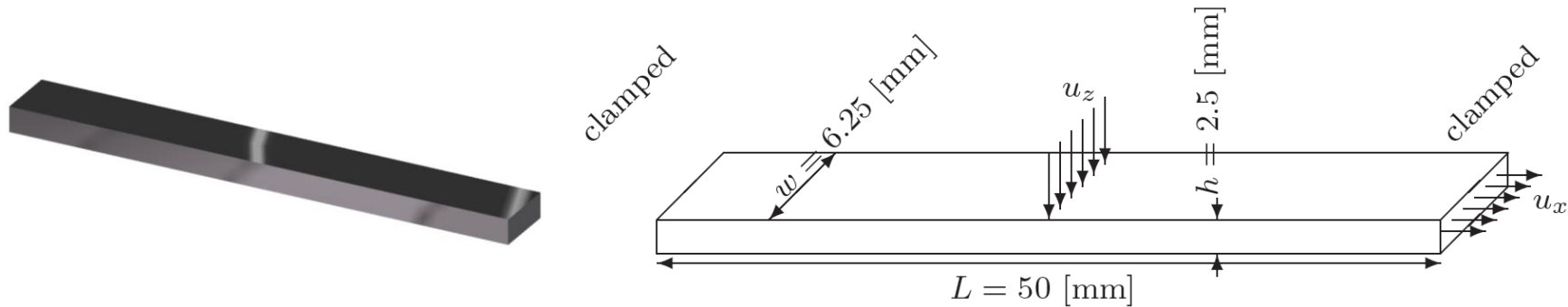
- The equivalent thicknesses become

$$h_I^{eq} = \frac{M_0}{h\sigma_I - N_0}$$

$$h_{II}^{eq} = \frac{M_0^T}{h\tau_{II} - T_0}$$

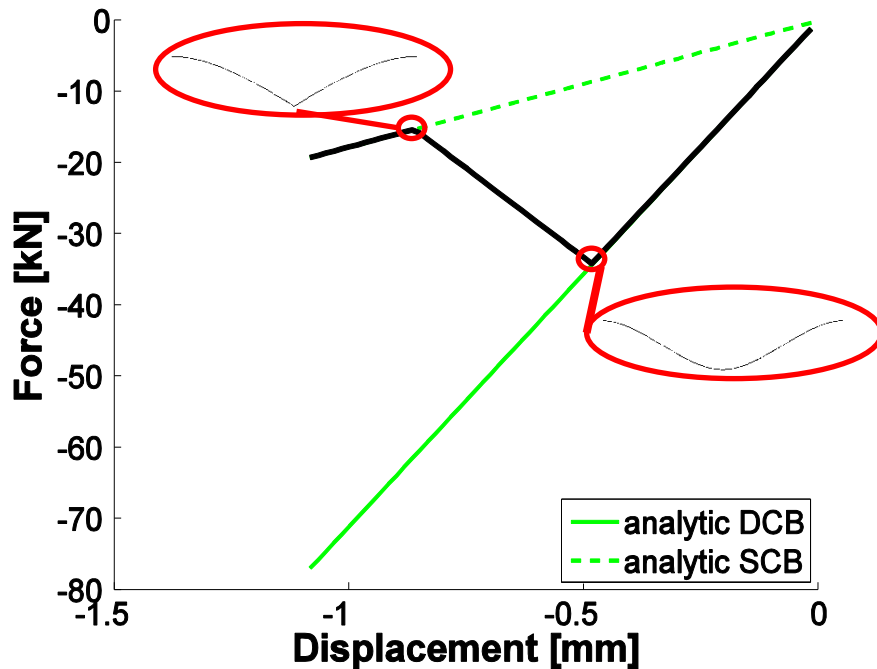
# Full-DG/ECL framework

- The transition between uncracked to fully cracked body depends on  $\Delta E_{int}$ 
  - Double clamped elastic beam loaded in a quasi-static way

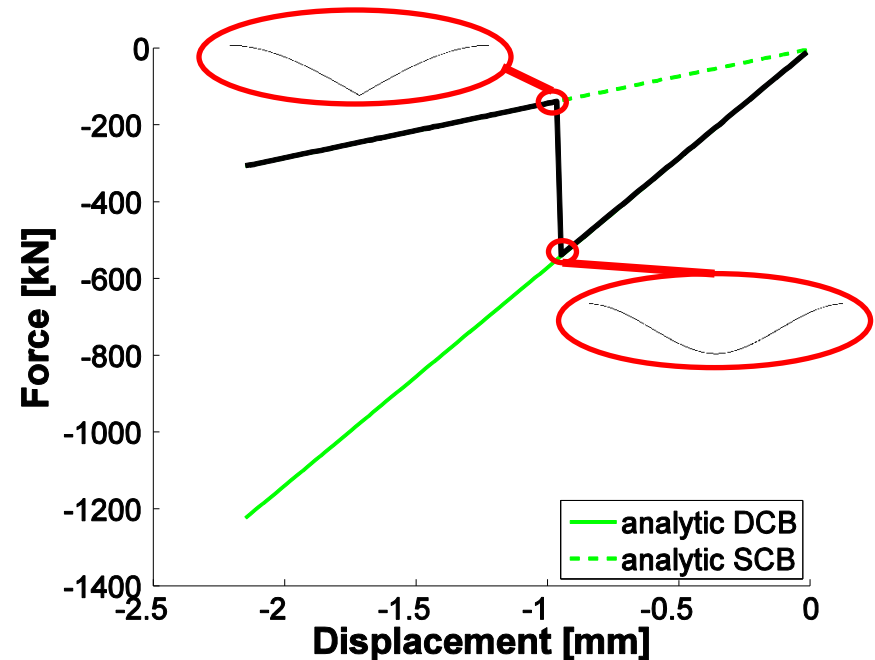


- The framework can model stable/unstable crack propagation
  - Geometry effect (no pre-strain)

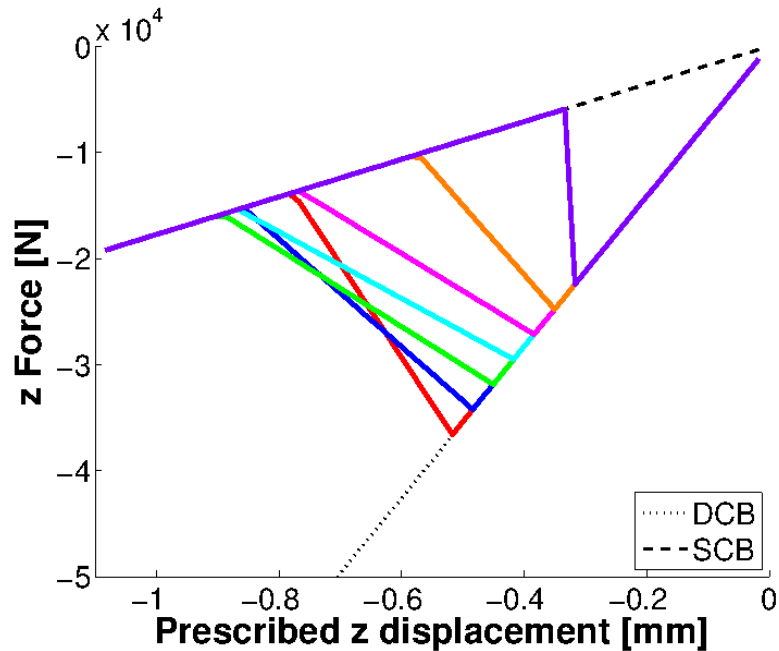
Stable transition  
 $\Delta E_{int}(u_z) < hGc$



Unstable transition  
 $\Delta E_{int}(u_z) > hGc$



- The energy released during fracture is always equal to  $hG_c$ 
  - Pre-strain effect



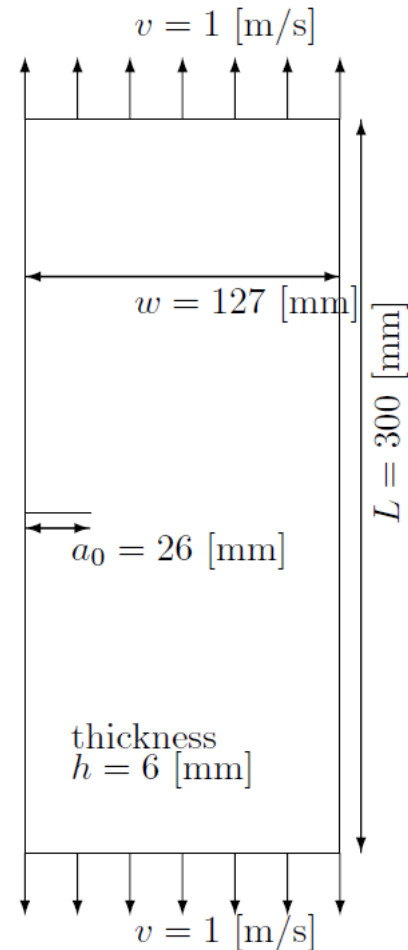
$hG_c = 22.00$

$u_{x,pres}$	$\eta_I$	$\Delta E_{int}$	$E_{released}$
$-2e^{-5}$	1,0692	14.82	21.98
0.	1	12.33	21.98
$2e^{-5}$	0.93	11.39	21.98
$4e^{-5}$	0.86	11.99	21.98
$6e^{-5}$	0.79	14.11	21.98
$8e^{-5}$	0.72	17.76	21.99
$10e^{-4}$	0.66	22.95	--

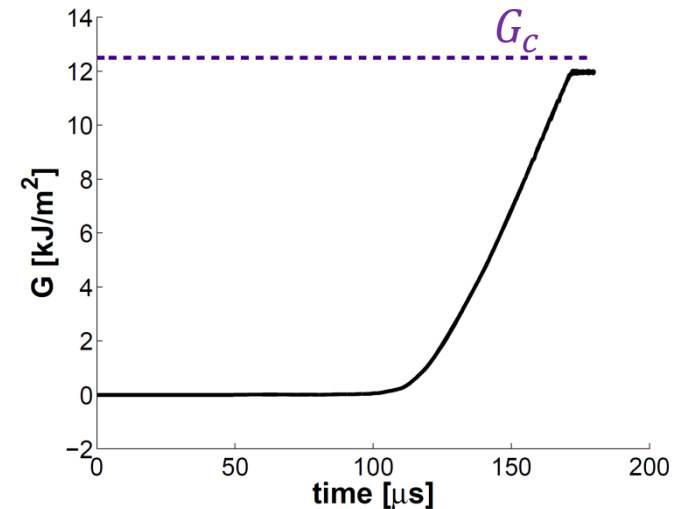
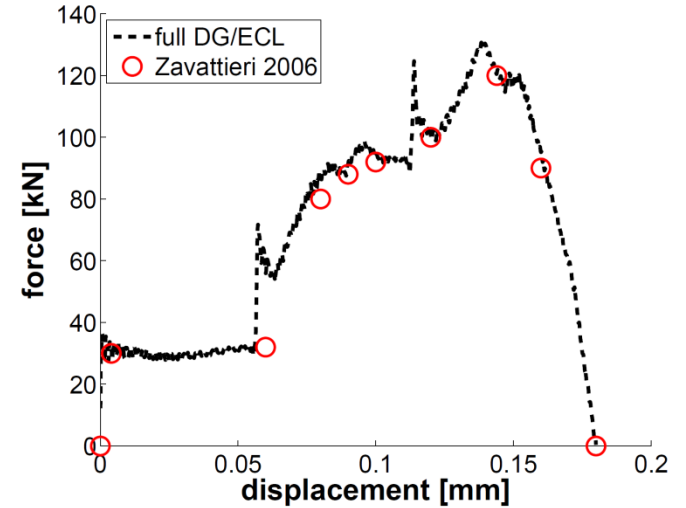
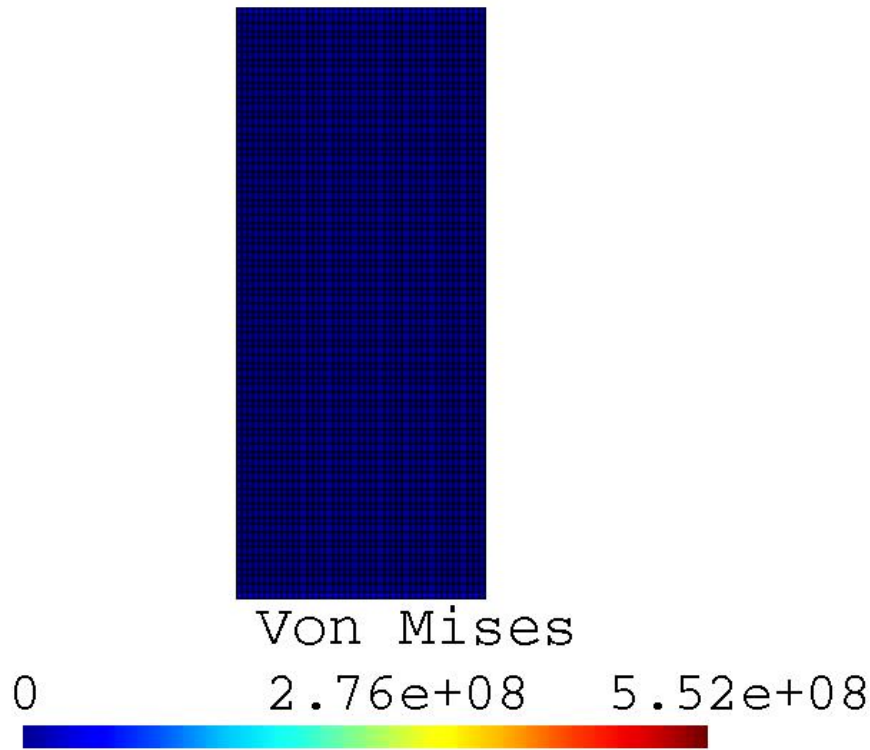
Pre-strain ↓

**> 22.00**  
**Unstable**

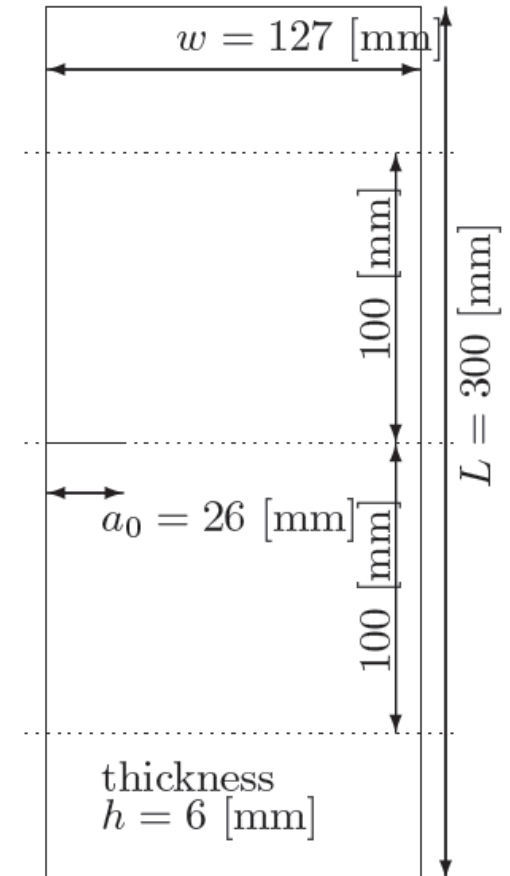
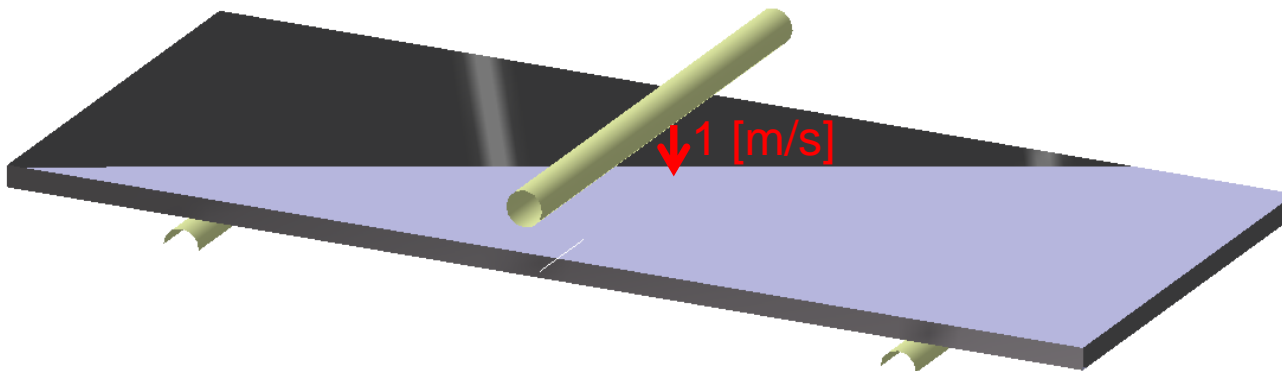
- A benchmark with a dynamic crack propagation
  - A single edge notched elastic plate dynamically loaded



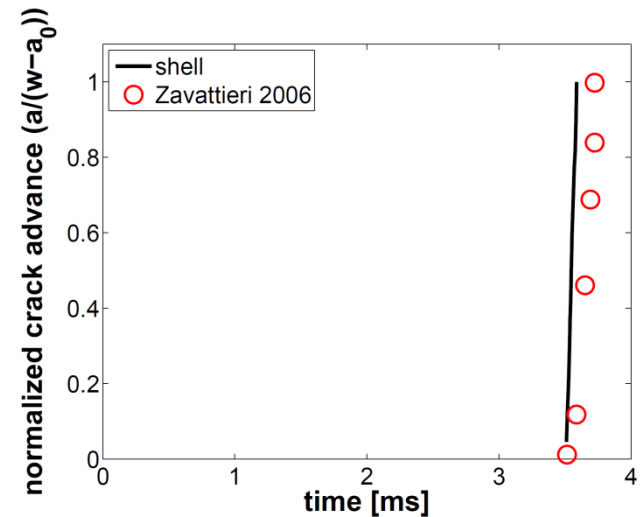
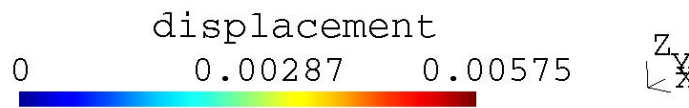
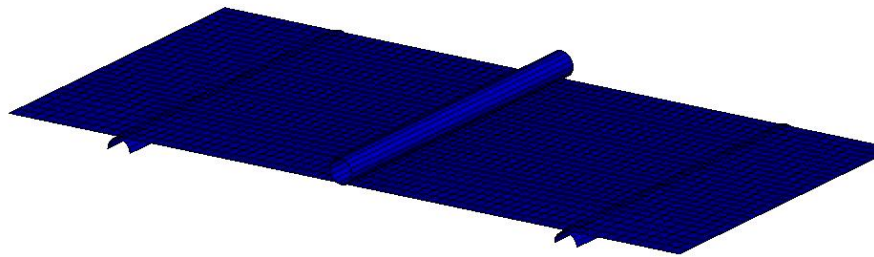
- The energy released in a dynamic crack propagation is equal to  $hG_c$ 
  - Compare results to the literature  
*[Zavattieri jam2006]*



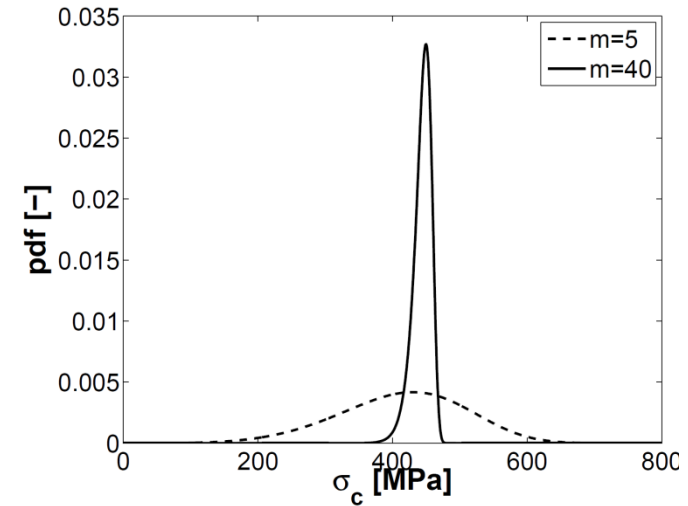
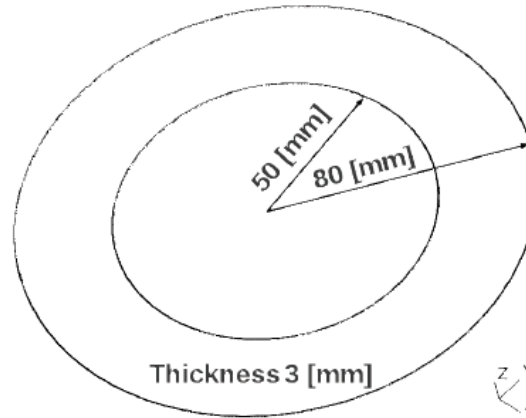
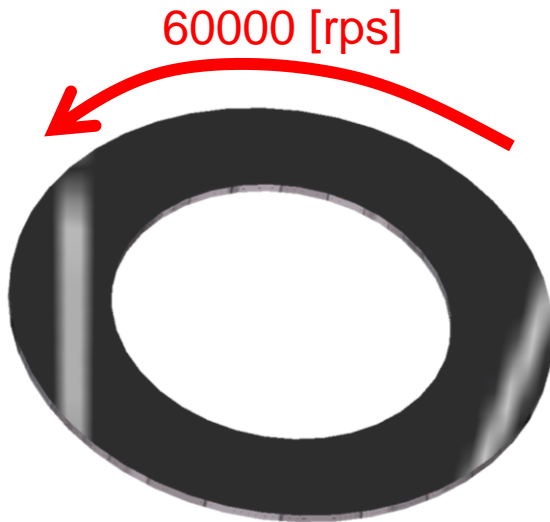
- A benchmark involving contact
  - A single edge notched elastic plate impacted by a rigid cylinder



- The crack propagates correctly even if there is (rigid) contact
  - Results are compared to the literature [*Zavattieri jam2006*]



- A benchmark to investigate the fragmentation
  - Elastic plate ring loaded by a centrifugal force



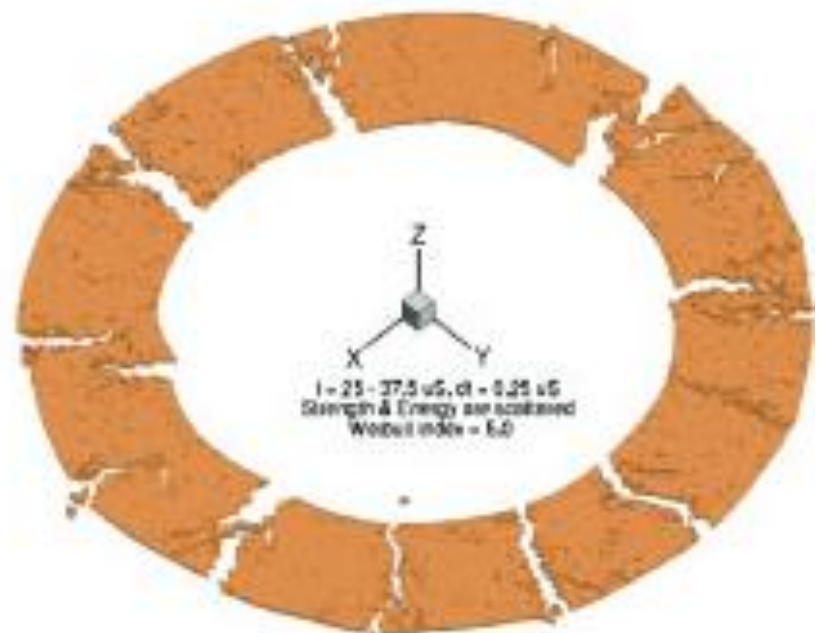
# Full-DG/ECL framework

- Fragmentation phenomena can also be studied by the full-DG/ECL framework
  - Results are compared with the literature [*Zhou et al ijmme2004*]



displacement

0 0.00015 0.0003



[*Zhou et al ijmme2004*]

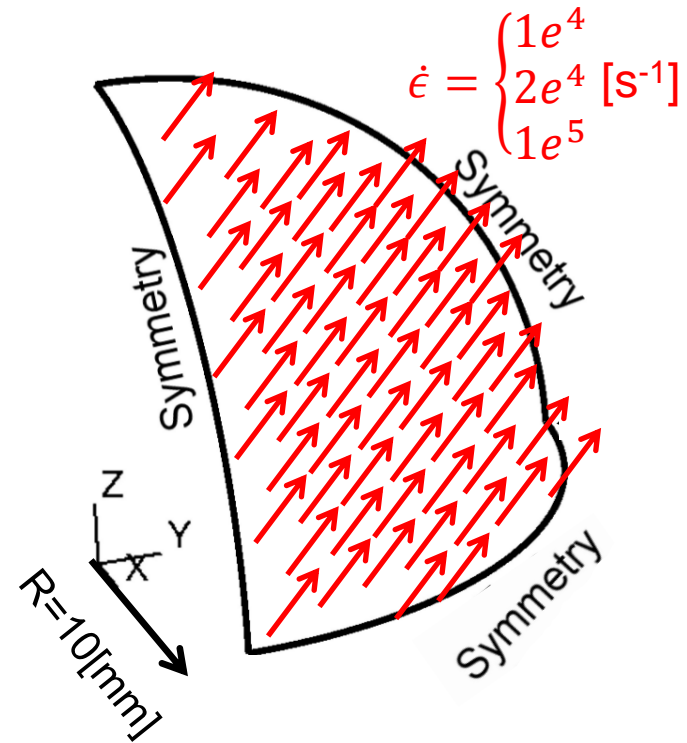
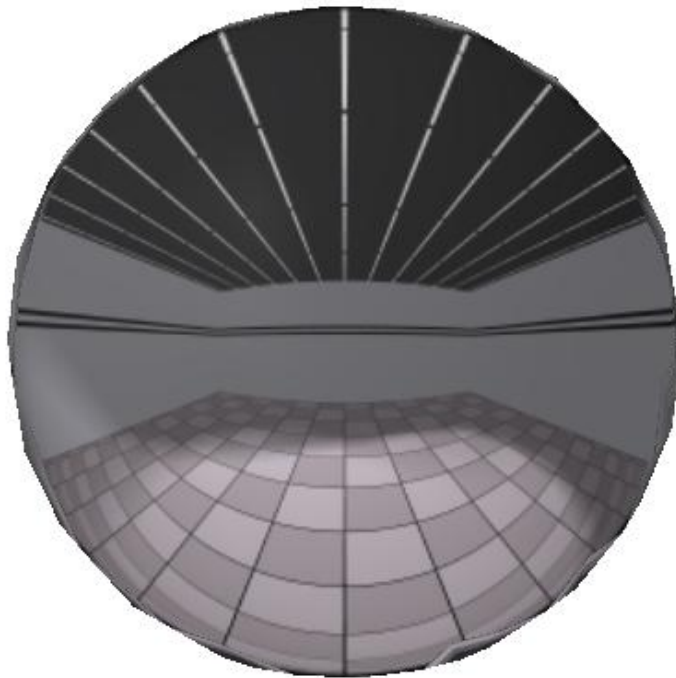
# Plan

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- Develop a discontinuous Galerkin method for thin bodies
  - Beam elements (1.5D case)
  - Shell elements (2.5D case)
- Discontinuous Galerkin / Extrinsic Cohesive law framework
  - Develop a suitable cohesive law for thin bodies
- Applications
  - Fragmentations, crack propagations under blast loadings

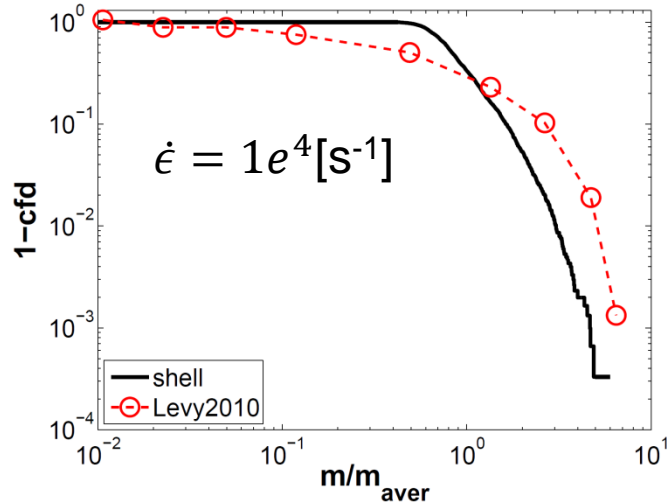
# Applications of the DG/ECL framework

- Application to the dynamic fragmentation of a sphere
  - Elastic sphere under radial uniform expansion

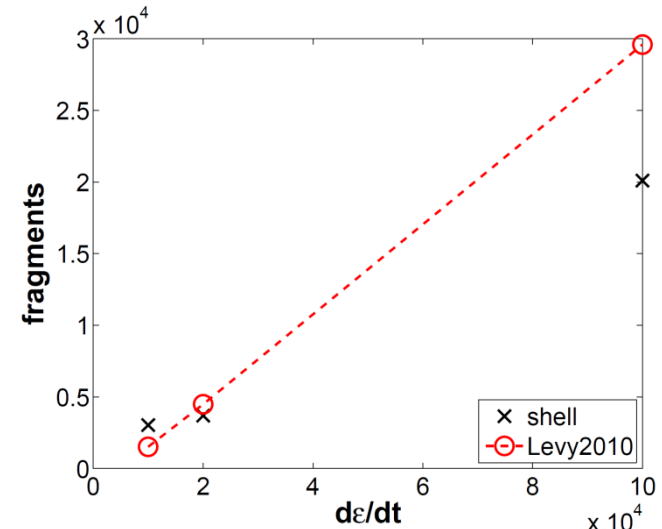
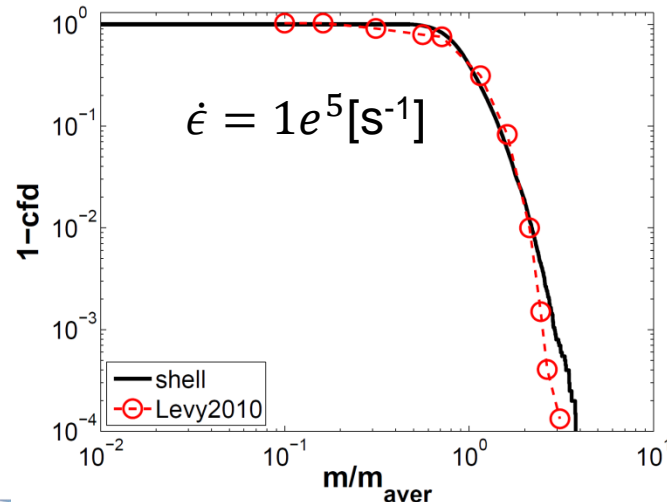
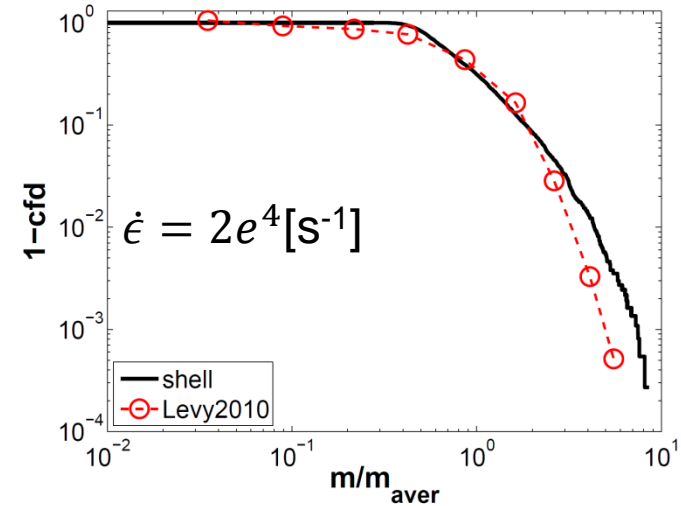


# Applications of the DG/ECL framework

- The distribution of fragments and the number of fragments are in agreement with the literature [*Levy EPFL2010*]
  - Levy uses 3D elements

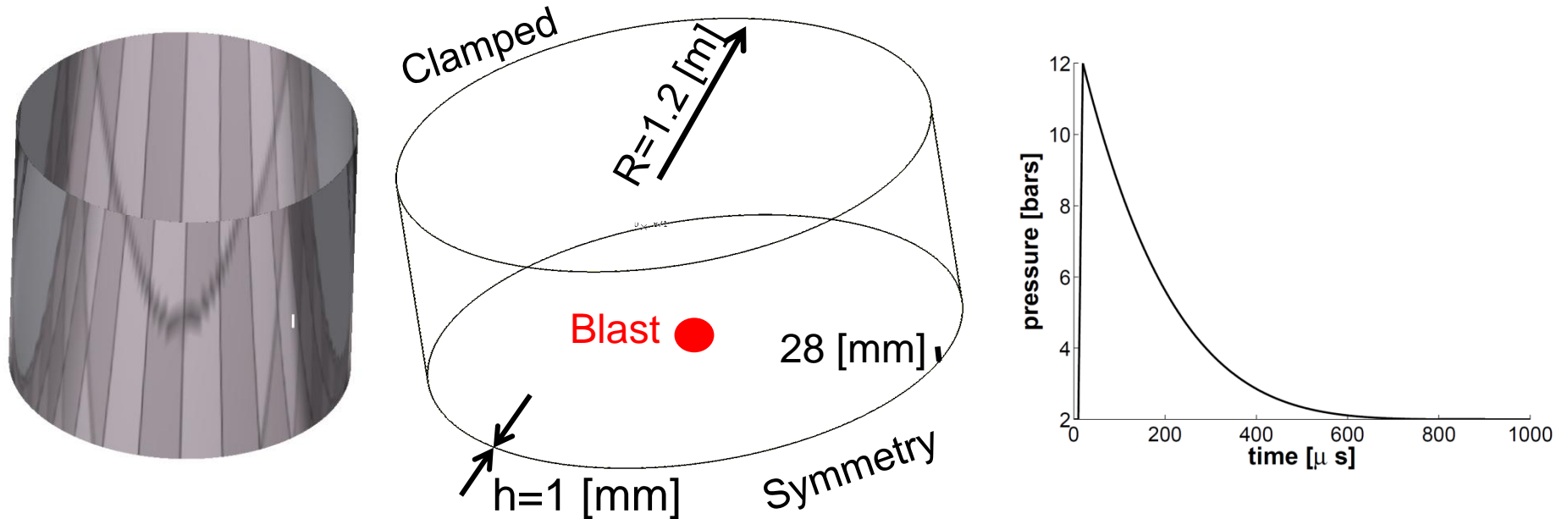


$\dot{\epsilon} = 1e^4 [s^{-1}]$   
 2 588 265 Dofs  
 $\pm 48h$  on 32 cpus  
 ( $\pm 17h$  for  $\dot{\epsilon} = 1e^5 [s^{-1}]$ )



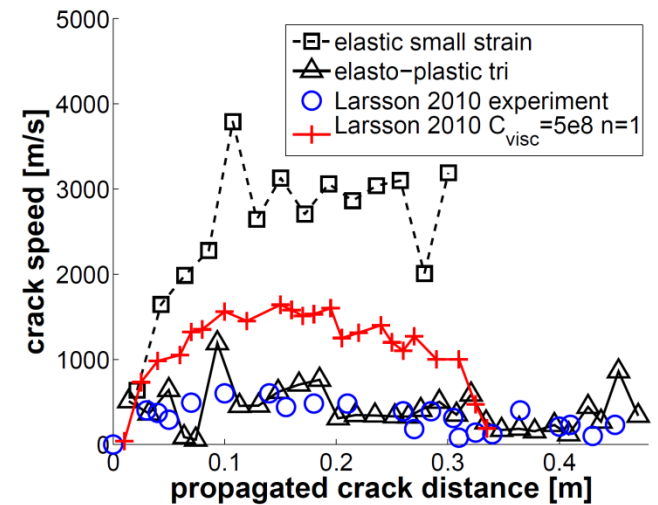
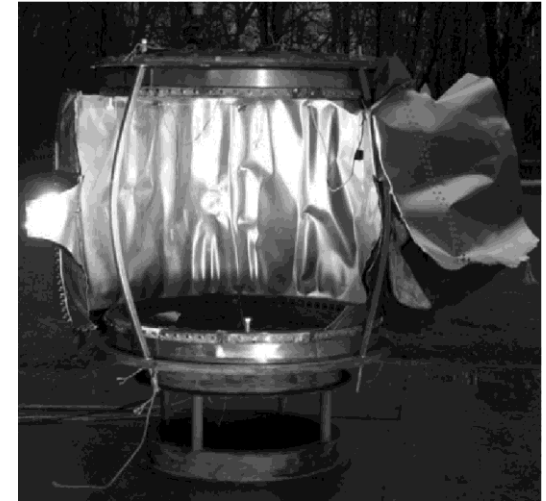
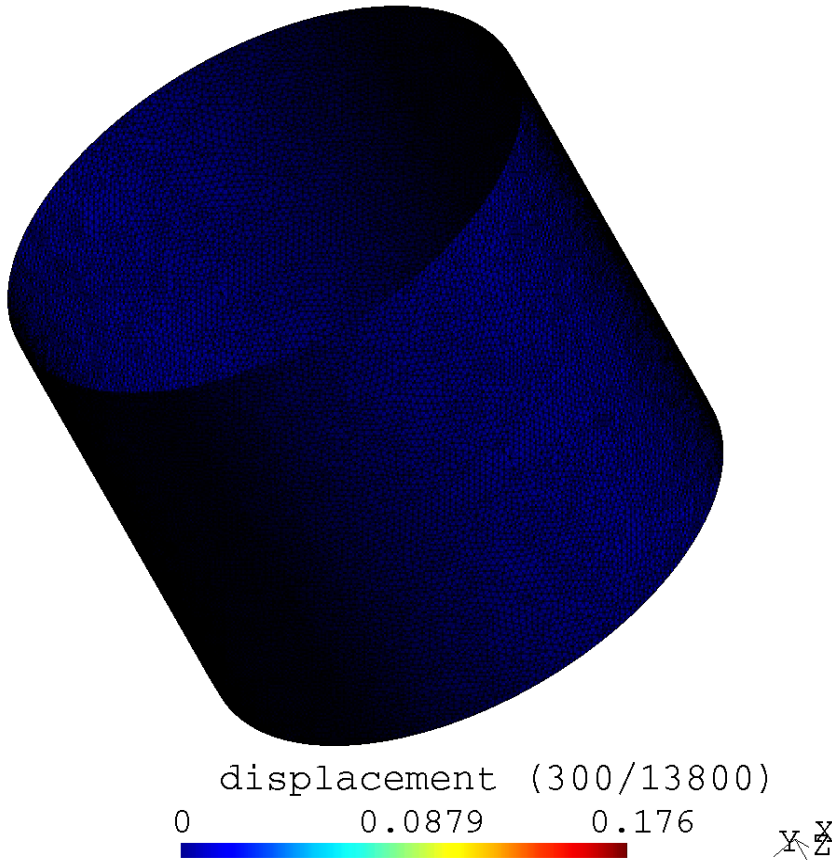
# Applications of the DG/ECL framework

- Blast of an axially notched elasto-plastic cylinder (large deformations)



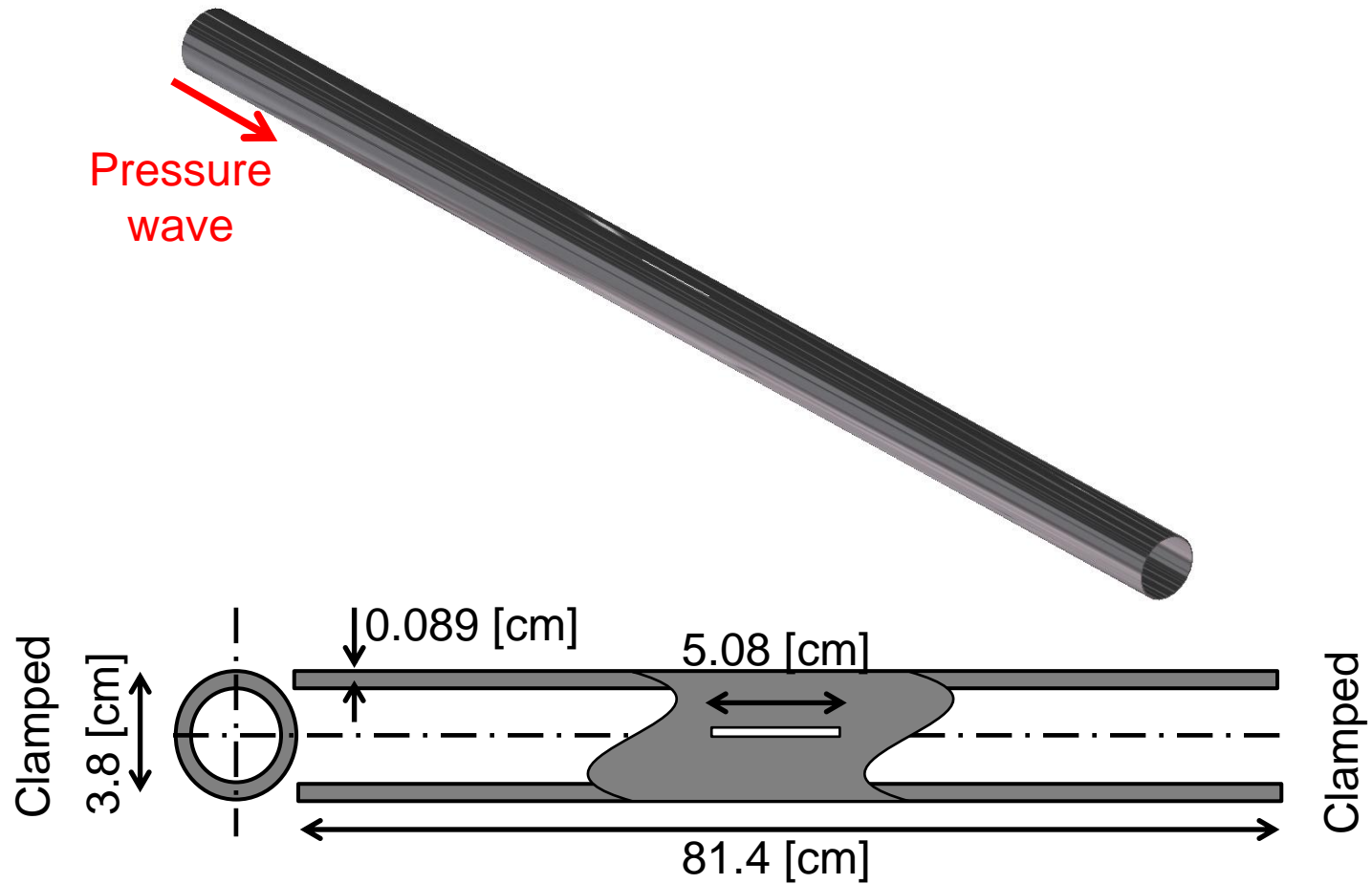
# Applications of the DG/ECL framework

- Accounting for plasticity allows capturing the crack speed
  - Compare with the literature [*Larson et al ijmme2011*]



# Applications of the DG/ECL framework

- Pressure wave passes through an axially notched elasto-plastic pipe (large deformations)

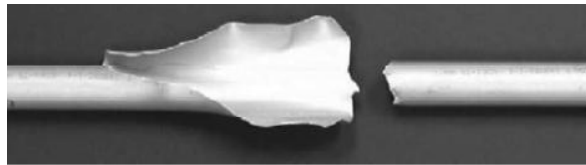


# Applications of the DG/ECL framework

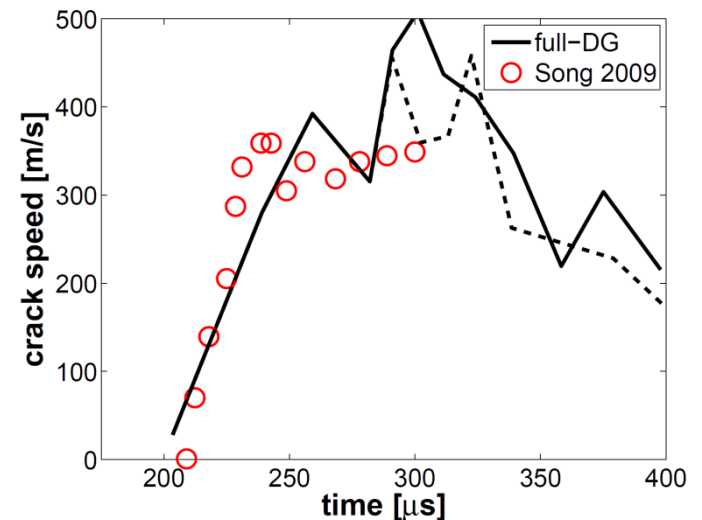
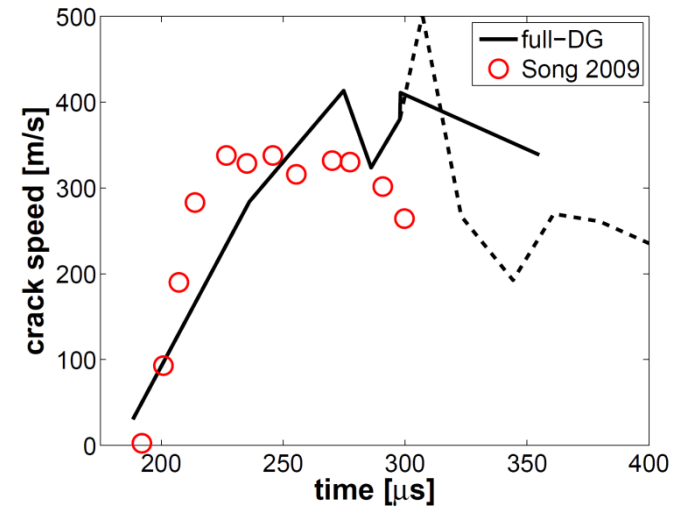
- Crack path and speed are well captured by the framework

– Compare with the literature

[*Song et al jam2009*]



224 256 Dofs  
±21 h on 12 cpus



# Conclusions

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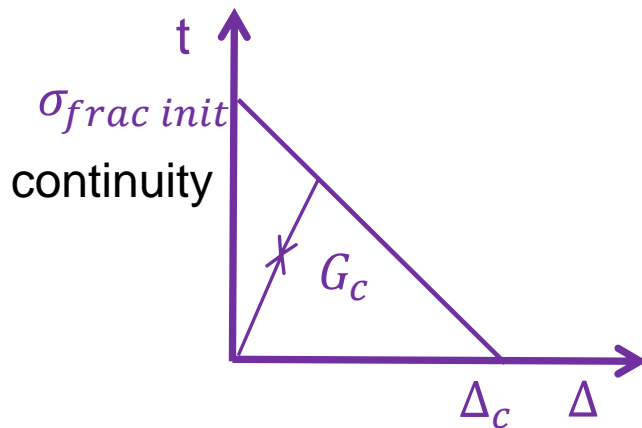
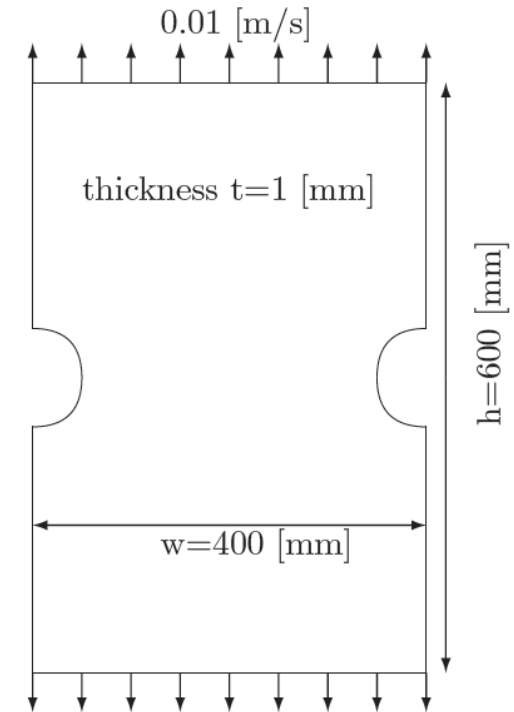
- Full-DG / ECL framework allows accounting for fracture in dynamic simulations of thin bodies
  - Crack propagation as well as fragmentation
  - Recourse to an elasto-plastic model is mandatory to capture crack speed
  - Affordable computational time for large models
- Main contributions
  - Full-DG model of linear Euler-Bernoulli beams and (non)-linear Kirchhoff-Love shell
  - Energetically consistent extrinsic cohesive law based on reduced stresses
  - Explicit Hulbert-Chung algorithm based on ghost elements (reduce MPI communication, independent of material law)

## Future work

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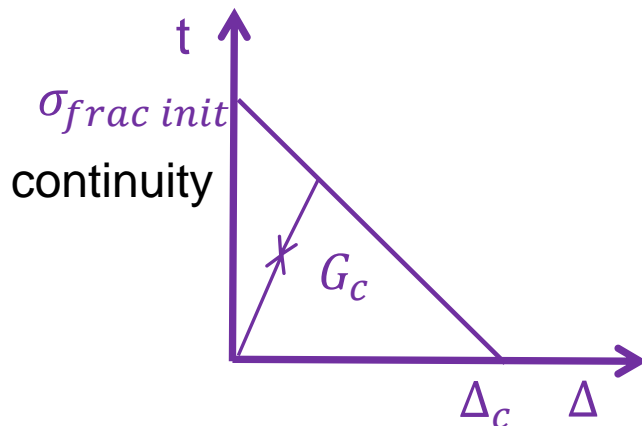
- Model the damage to crack transition by coupling a damage law with the full-DG/ECL framework
  - Replace the criterion based on an effective stress by a criterion based on the damage
  - Define the shape of the cohesive law

- An exploratory benchmark
  - Quasi-static (dynamic relaxation)
  - Linear damage theory
  - Fracture criterion  $D > D_c$
  - Cohesive shape

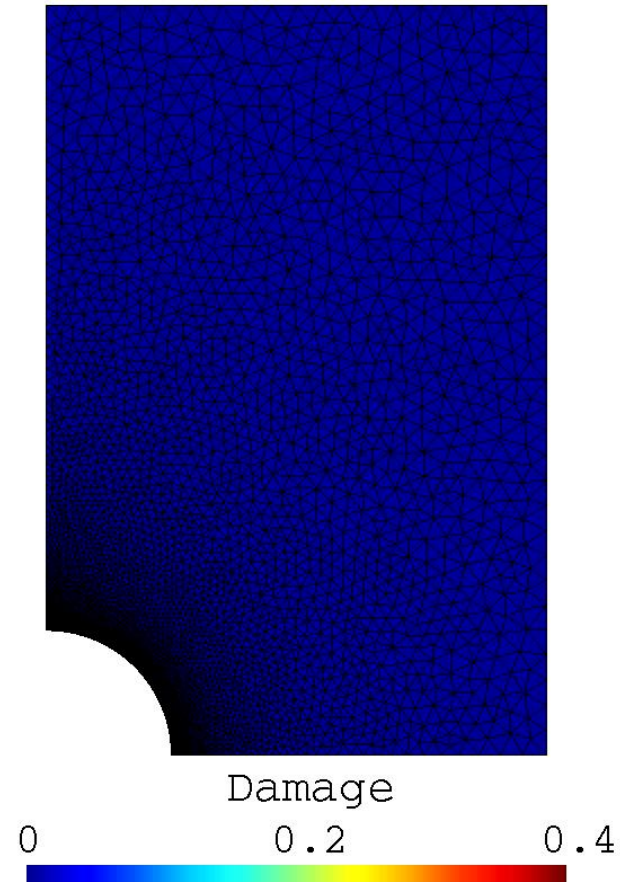


$G_c$  from the literature [Mazars et al ijss1996]

- The benchmark shows encouraging perspectives
  - Linear damage theory
  - Fracture criterion  $D > D_c$
  - Cohesive shape



$G_c$  from the literature [Mazars et al ijss1996]



## Future work

---

- The benchmark shows encouraging perspectives but many improvements are required
  - Non local damage model
  - Account for stress triaxiality (and out-of-plane shearing)
  - Shape of the cohesive law
  - ...

---

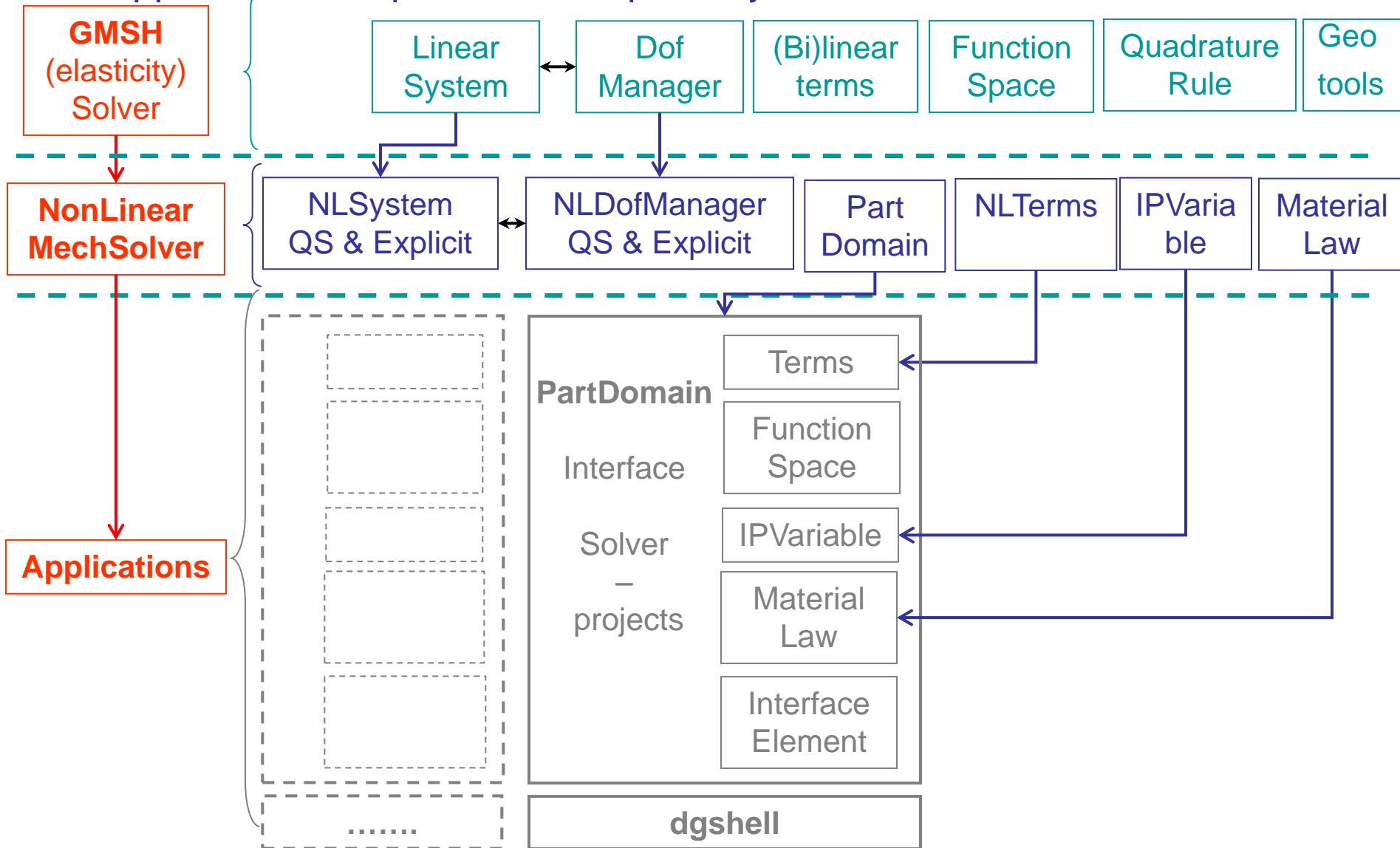
# Thank you for your attention

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# Appendix

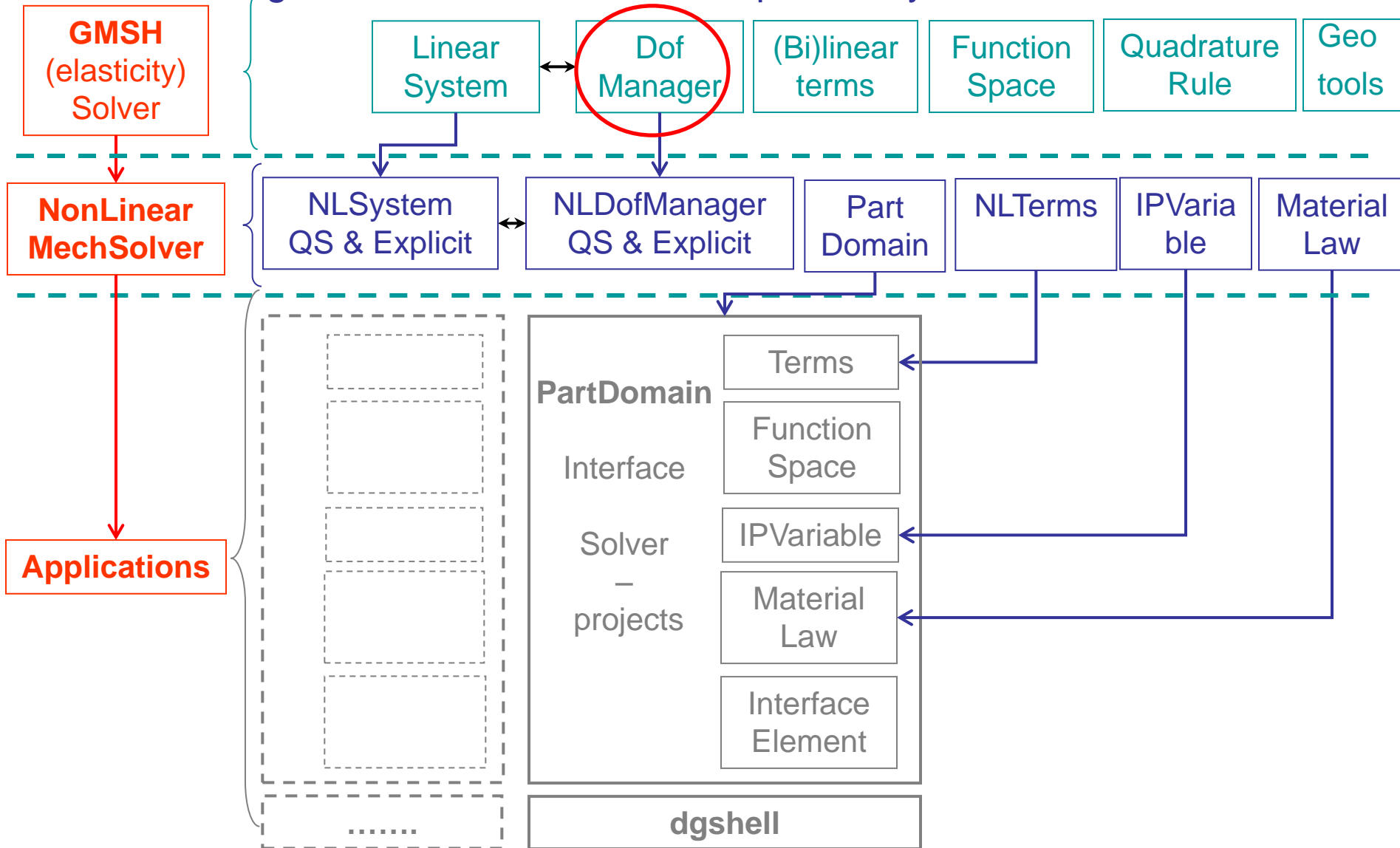
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- Application is implemented separately from the solver to be versatile



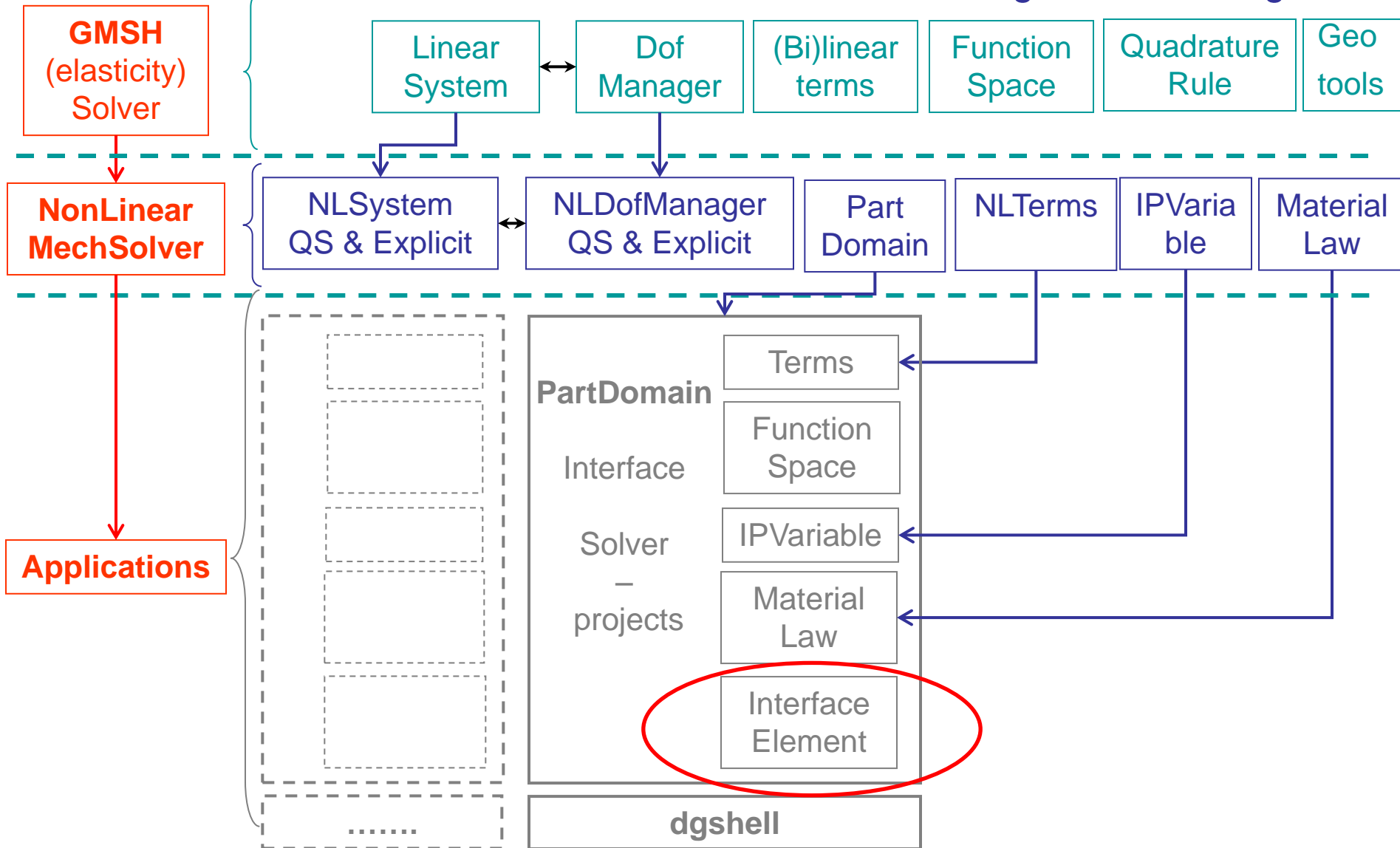
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- DofManager allows to define dof independently of the mesh



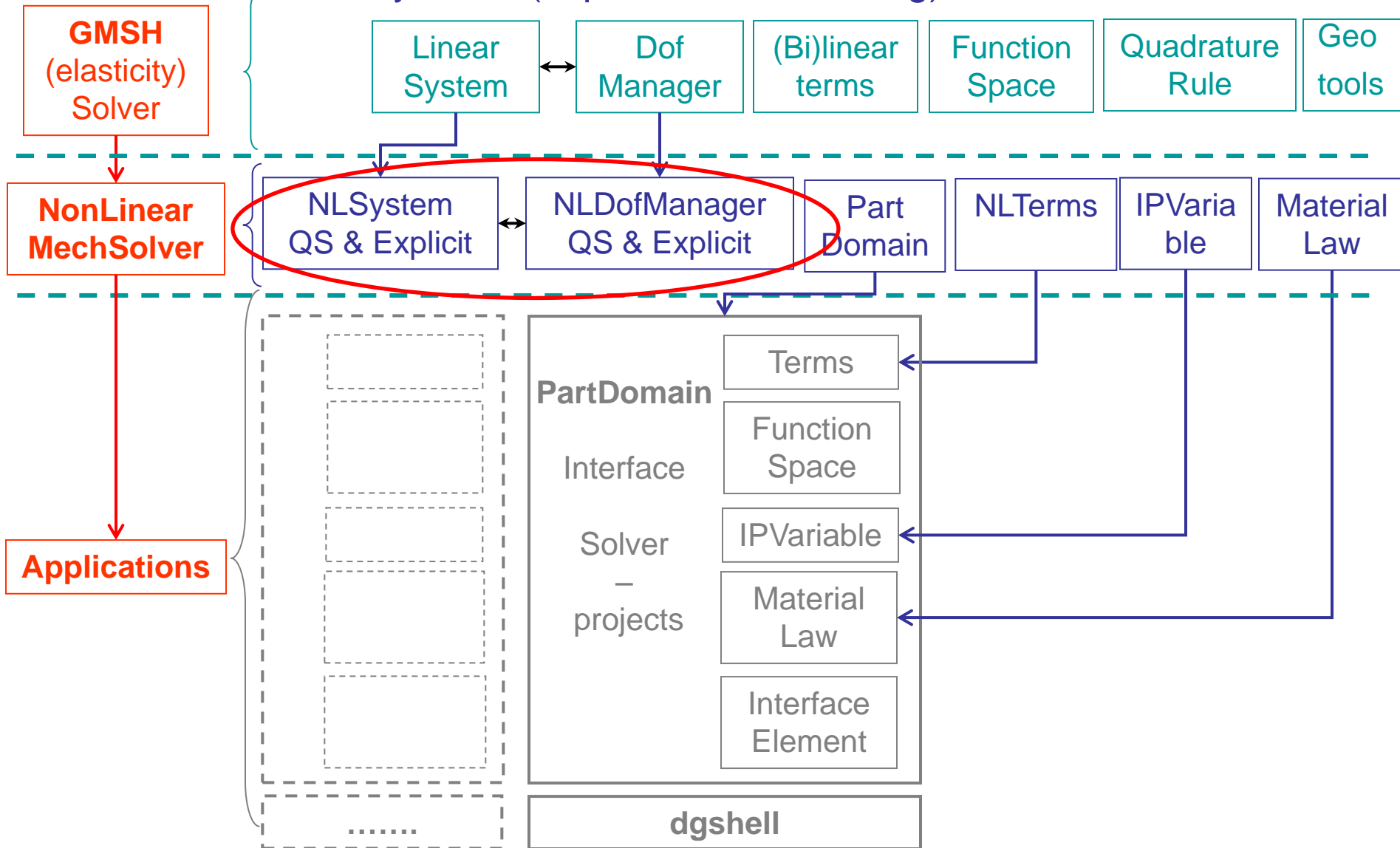
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- Continuous mesh is used as interface elements are generated in dgshell



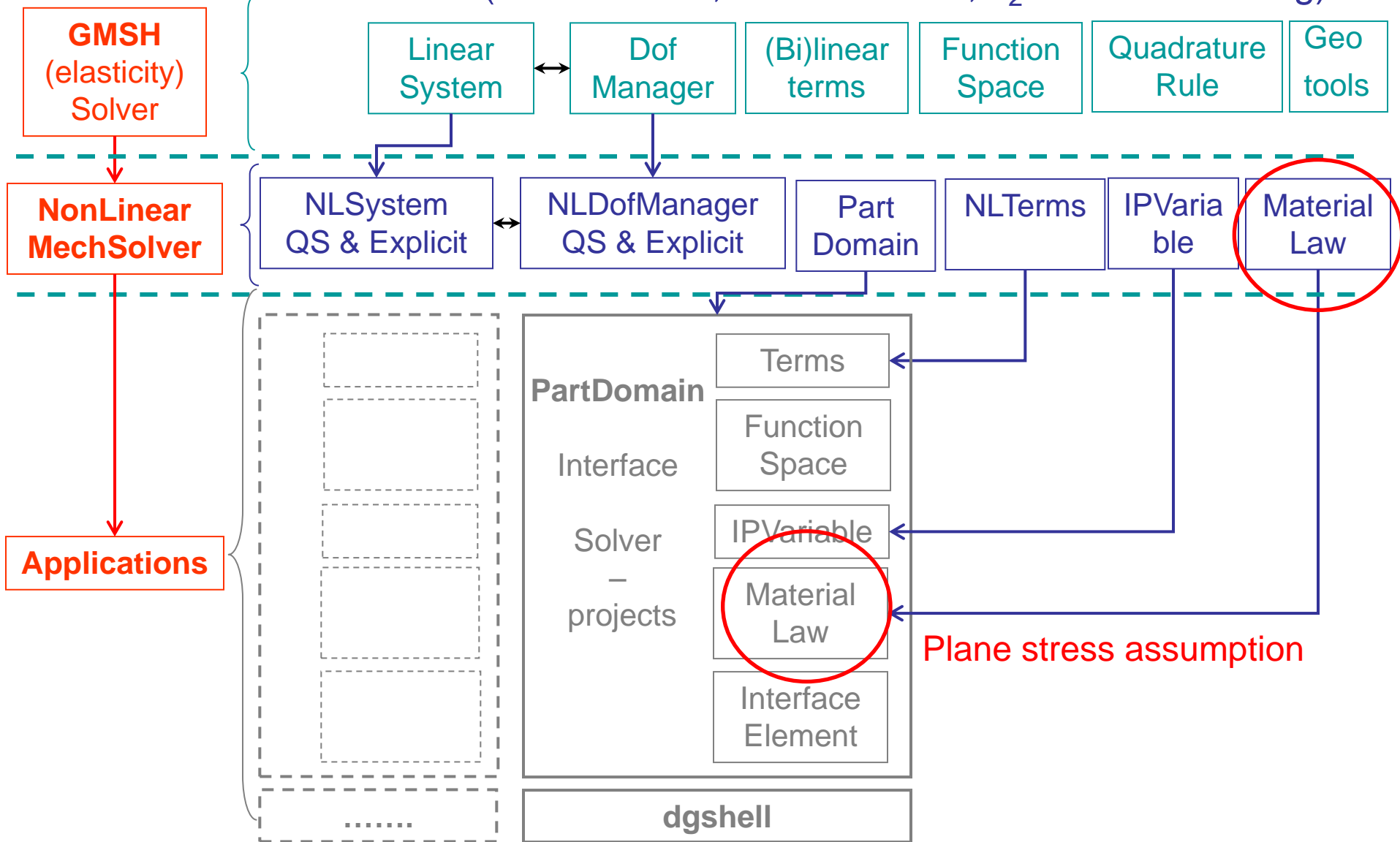
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- Quasi-static or dynamic (explicit Hulbert-Chung) schemes are available



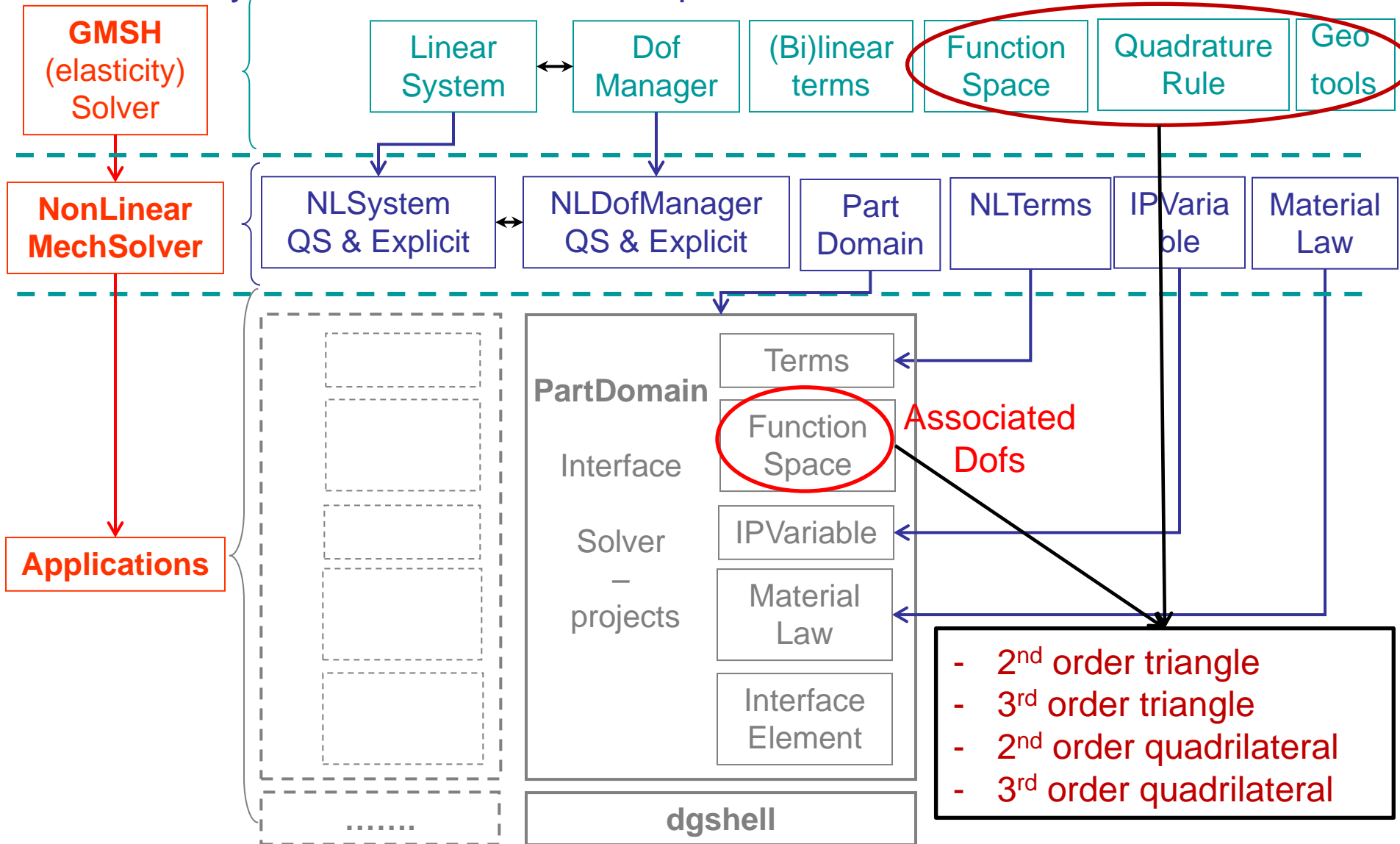
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- Different material law (elastic linear, neo-Hookean,  $J_2$ -linear hardening)



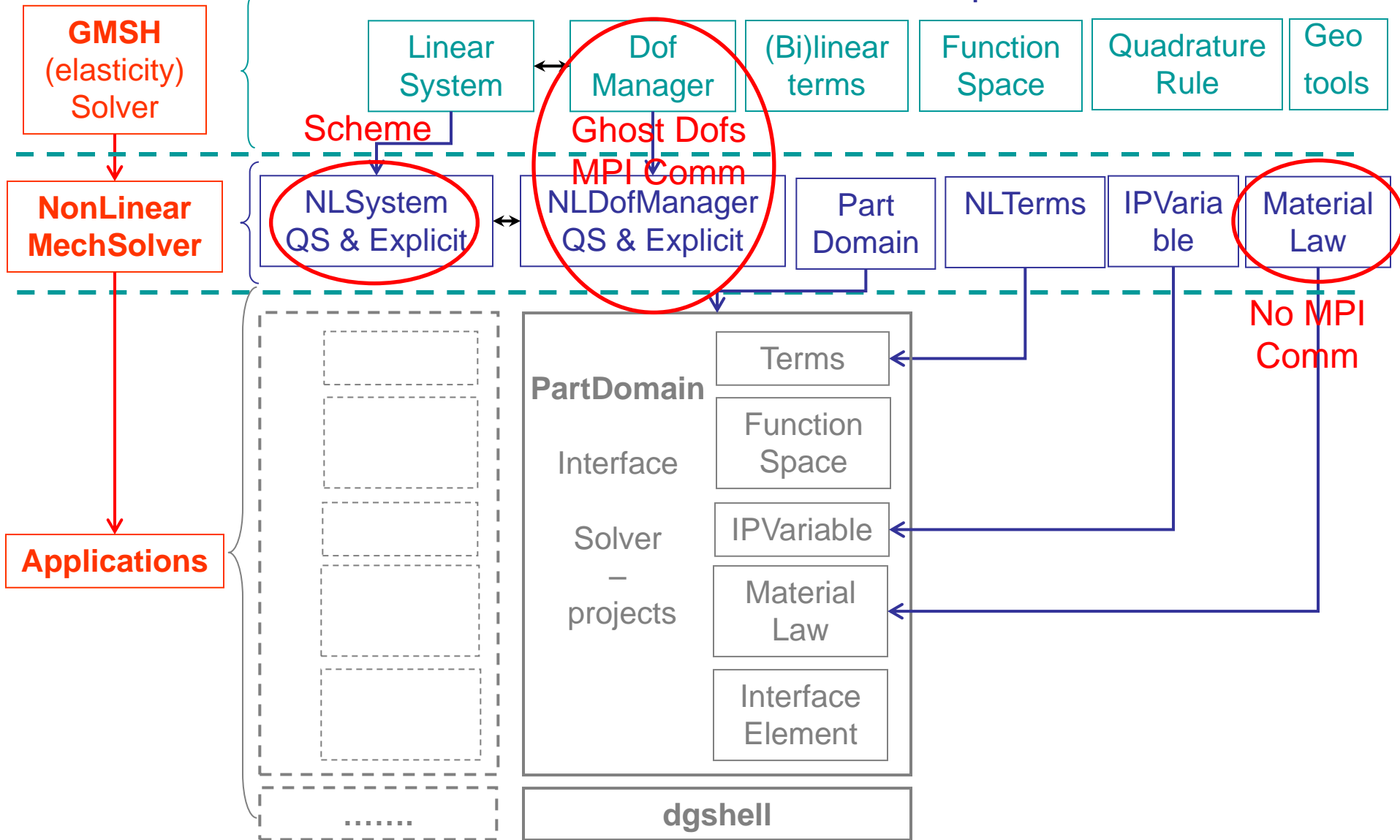
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- A library of 4 shell elements is implemented



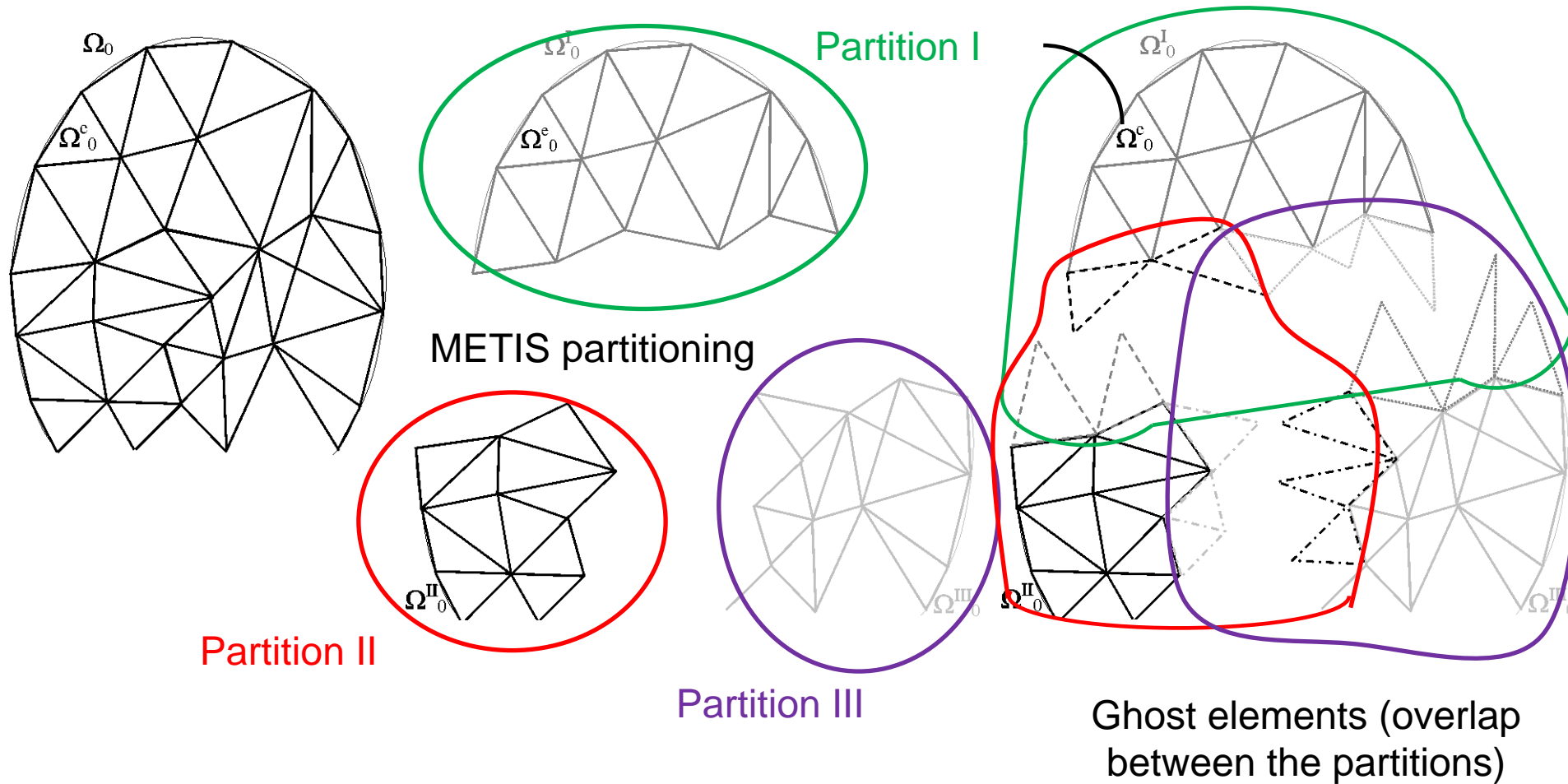
# Implementation of the full-DG formulation of Kirchhoff-Love shells

- Parallel scheme is based on Ghost element concept



# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- Ghost elements of a partition are the elements of other partitions which have a common interface with this partition



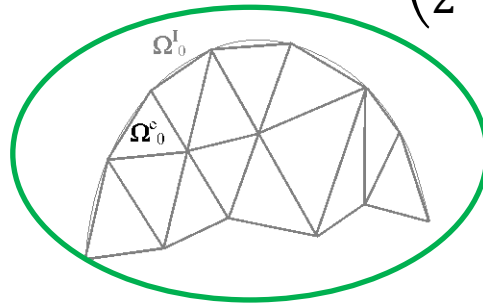
# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- Solve for the dofs of the elements of the partition linked to the processor
  - $M$  is the diagonalized mass matrix

$$\ddot{\mathbf{x}}^{n+1} = \frac{1}{1 - \alpha_M} \mathbf{M}^{-1} \cdot [\mathbf{F}_{ext}^n - \mathbf{F}_{int}^n] - \frac{\alpha_M}{1 - \alpha_M} \ddot{\mathbf{x}}^n$$

$$\dot{\mathbf{x}}^{n+1} = \dot{\mathbf{x}}^n + \Delta t(1 - \gamma)\ddot{\mathbf{x}}^n + \Delta t\gamma\ddot{\mathbf{x}}^{n+1}$$

$$\mathbf{x}^{n+1} = \mathbf{x}^n + \Delta t \dot{\mathbf{x}}^n + \Delta t^2 \left( \frac{1}{2} - \beta \right) \ddot{\mathbf{x}}^n + \Delta t^2 \beta \ddot{\mathbf{x}}^{n+1}$$



Processor I

$\mathbf{x}_{partition I}$ ,

$\dot{\mathbf{x}}_{partition I}$ ,

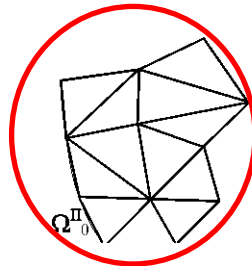
$\ddot{\mathbf{x}}_{partition I}$

Processor II

$\mathbf{x}_{partition II}$ ,

$\dot{\mathbf{x}}_{partition II}$ ,

$\ddot{\mathbf{x}}_{partition II}$

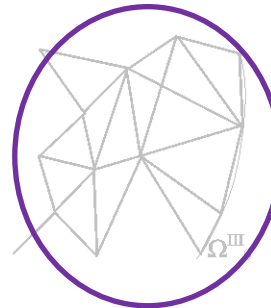


Processor III

$\mathbf{x}_{partition III}$ ,

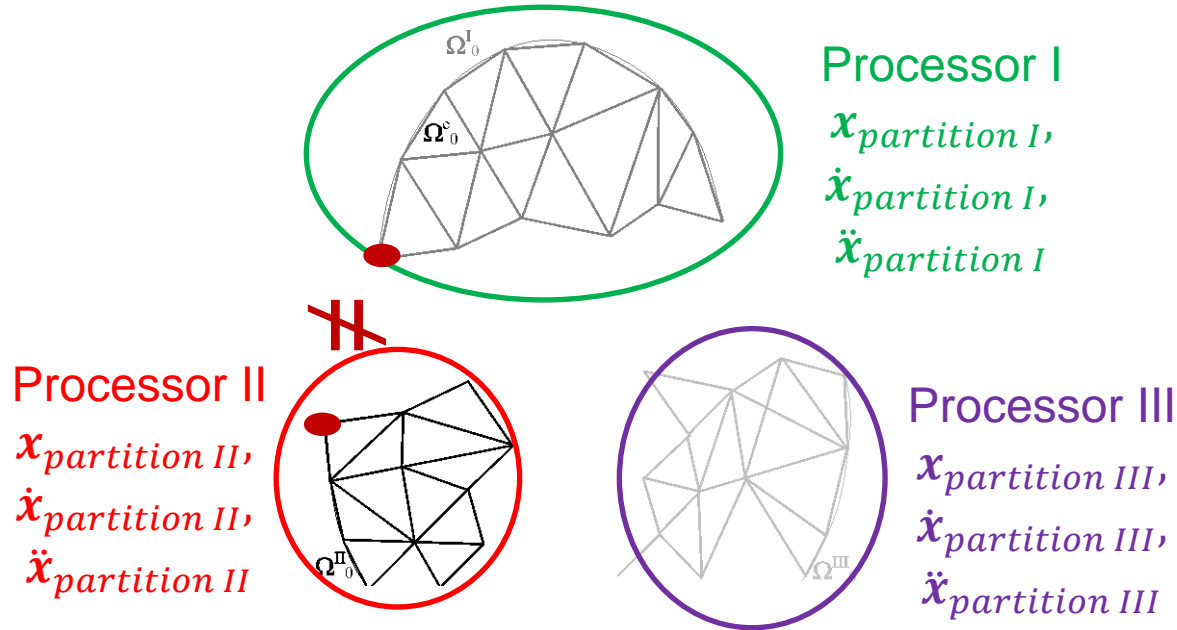
$\dot{\mathbf{x}}_{partition III}$ ,

$\ddot{\mathbf{x}}_{partition III}$



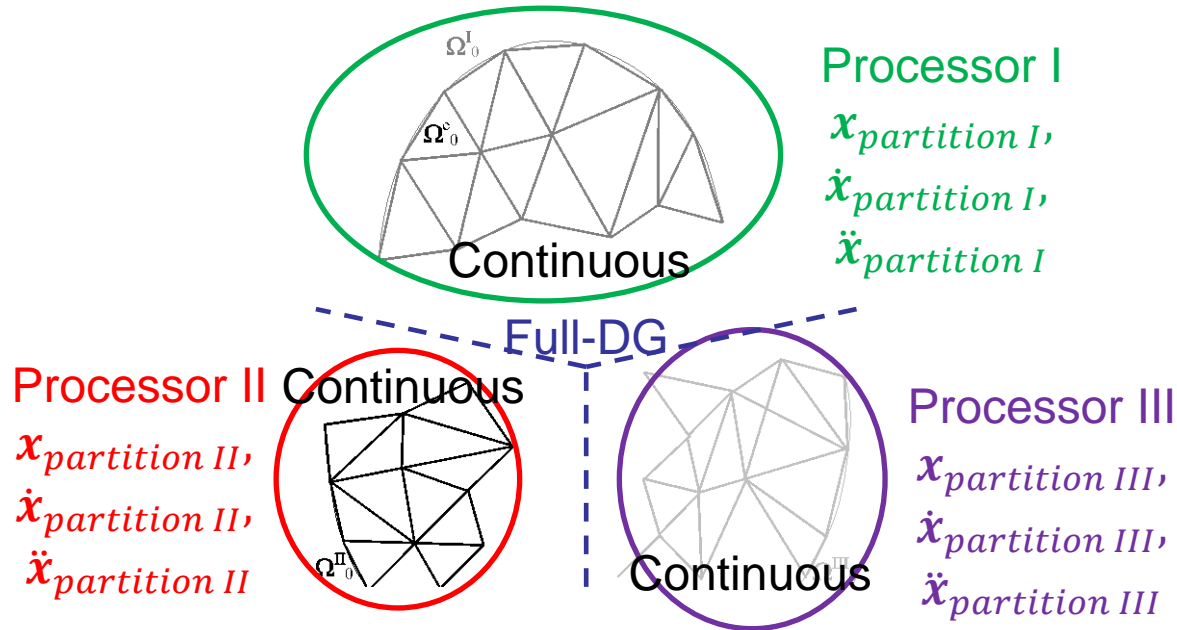
# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- The elements have to be discontinuous between partition



# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

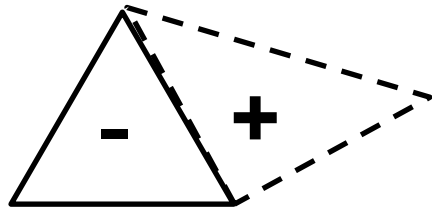
- Recourse to the full-DG formulation between partitions to ensure continuity between them
  - Extra dofs are only inserted between partitions



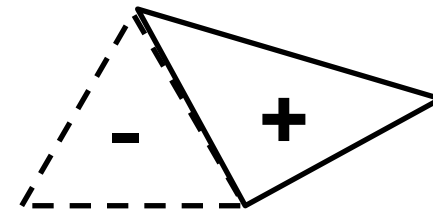
- Interface elements have to be computed

# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- Interface elements are computed on each partition (using Ghost elements)
  - Processor I
  - Processor II



$$\mathbf{F}_{int} = \begin{bmatrix} \mathbf{F}_{int}^- \\ \mathbf{F}_{int}^+ \end{bmatrix}$$

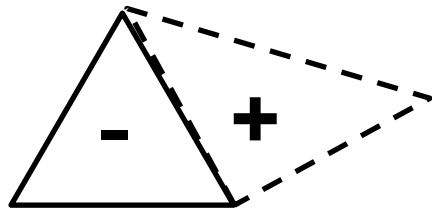


$$\mathbf{F}_{int} = \begin{bmatrix} \mathbf{F}_{int}^- \\ \mathbf{F}_{int}^+ \end{bmatrix}$$

# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- Only the part of  $\mathbf{F}_{int}$  associated to the dofs of the partition is assembled

– Processor I

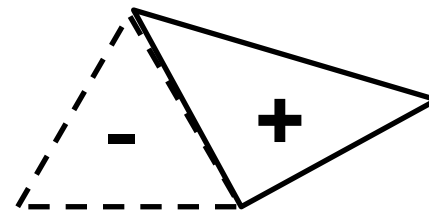


$$\mathbf{F}_{int} = \begin{bmatrix} \mathbf{F}_{int}^- \\ \mathbf{F}_{int}^+ \end{bmatrix}$$

Assembled

Explicit system  
(Partition I)

– Processor II



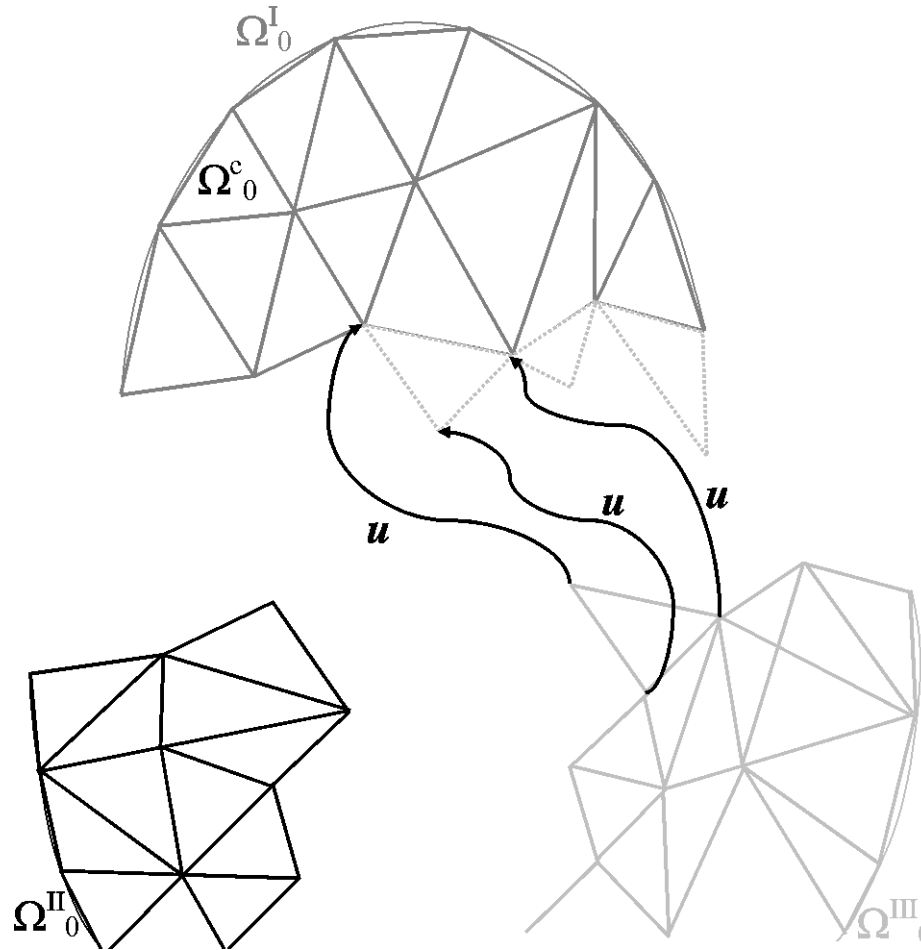
$$\mathbf{F}_{int} = \begin{bmatrix} \mathbf{F}_{int}^- \\ \mathbf{F}_{int}^+ \end{bmatrix}$$

Assembled

Explicit system  
(Partition II)

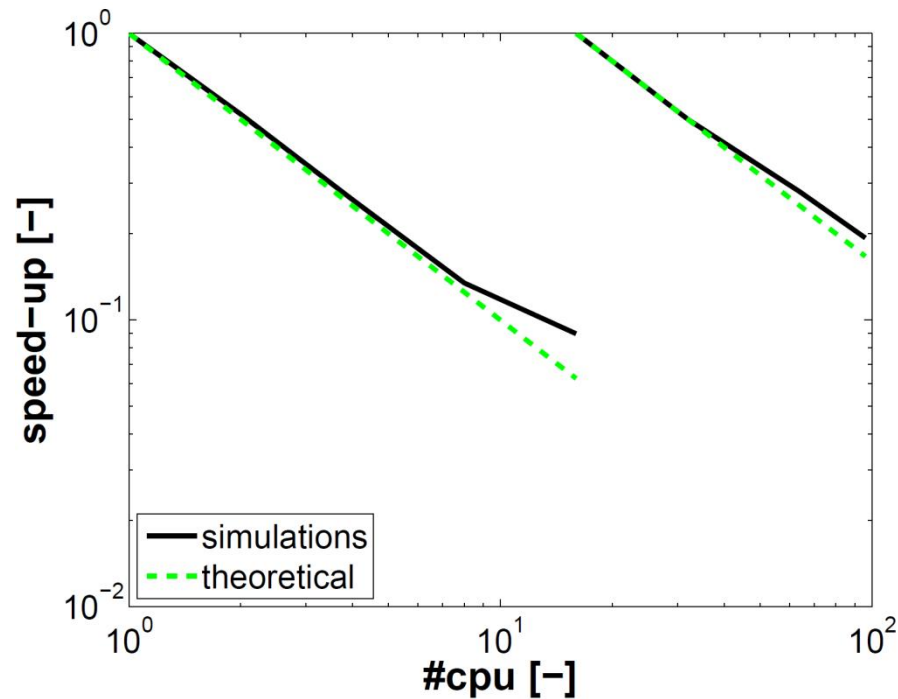
# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- Only one MPI communication is required by time step
  - Unknowns are exchanged before the computation of  $F_{int}$



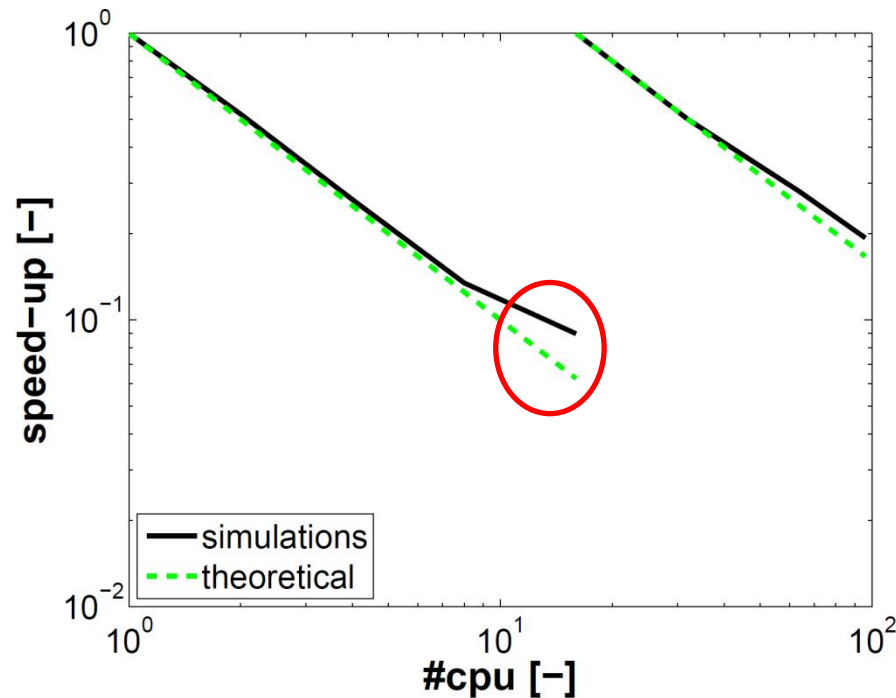
# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

- The parallel scheme is almost optimal
  - Theoretical speed-up = time n processor / time 1 processor = 1/n
  - Practically the speed-up is lower than expected (MPI communication)
  - Cylindrical panel benchmark on Nic3 (cluster with 8 cores 2.5Ghz per node)



# Parallel implementation of full-DG formulation of Kirchhoff-Love shells

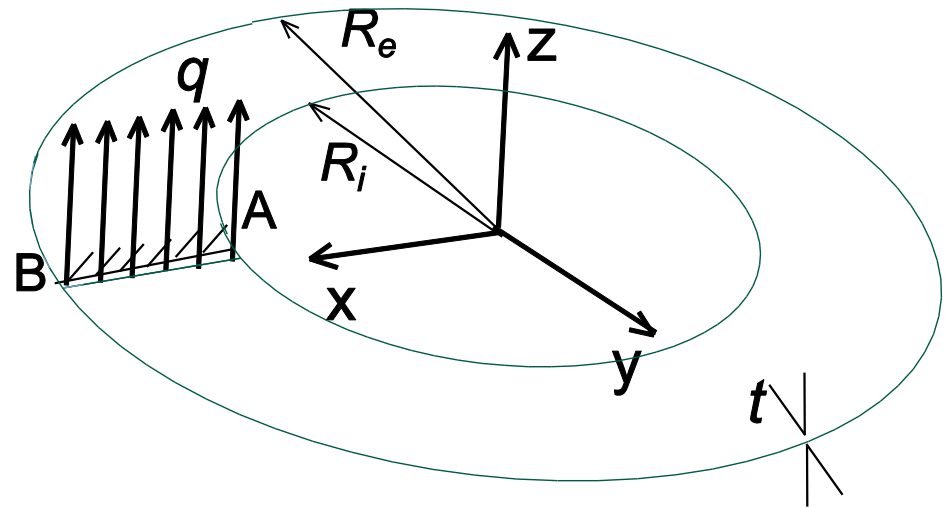
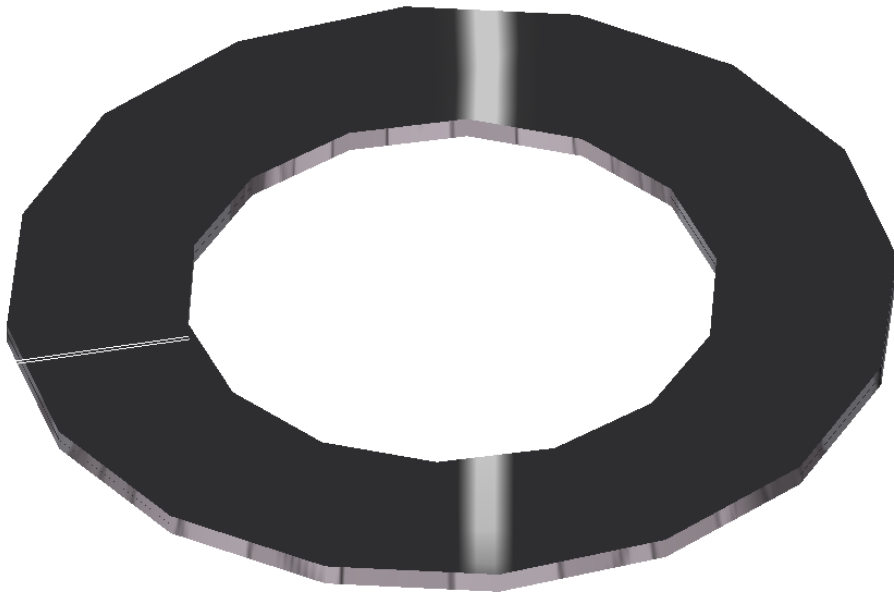
- The parallel scheme is almost optimal when the number of elements remains large compared to the number of interfaces (OK in practice)
  - Theoretical speed-up = time n processor / time 1 processor = 1/n
  - Practically the speed-up is lower than expected (MPI communication)
  - Cylindrical panel benchmark on Nic3 (cluster with 8 cores 2.5Ghz per node)



The number of interfaces is not negligible compared to the number of elements on each partition ( $\pm 1$  interface for 3 elements)

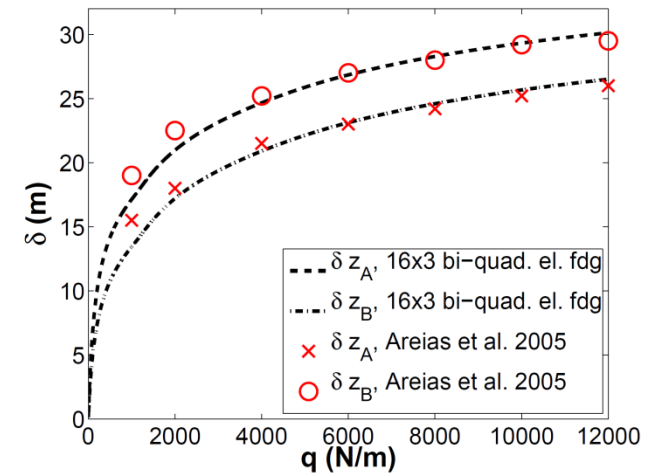
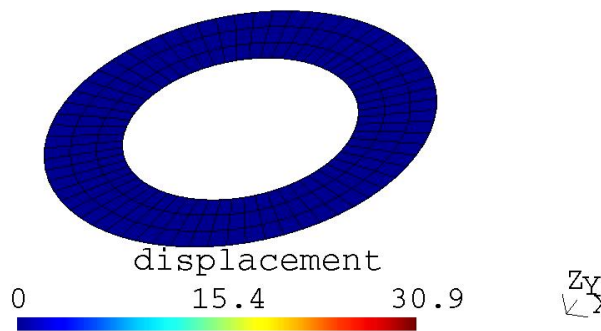
## Full-DG formulation extra benchmarks

- Neo-Hookean (elastic large deformations) plate ring loaded in a quasi-static way



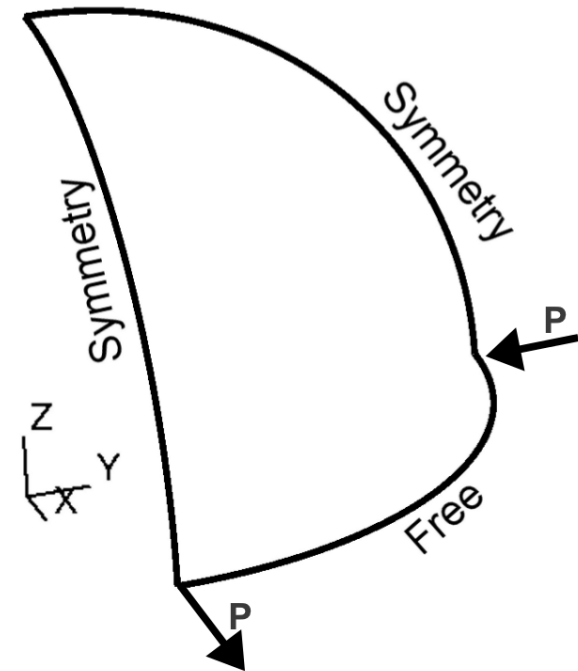
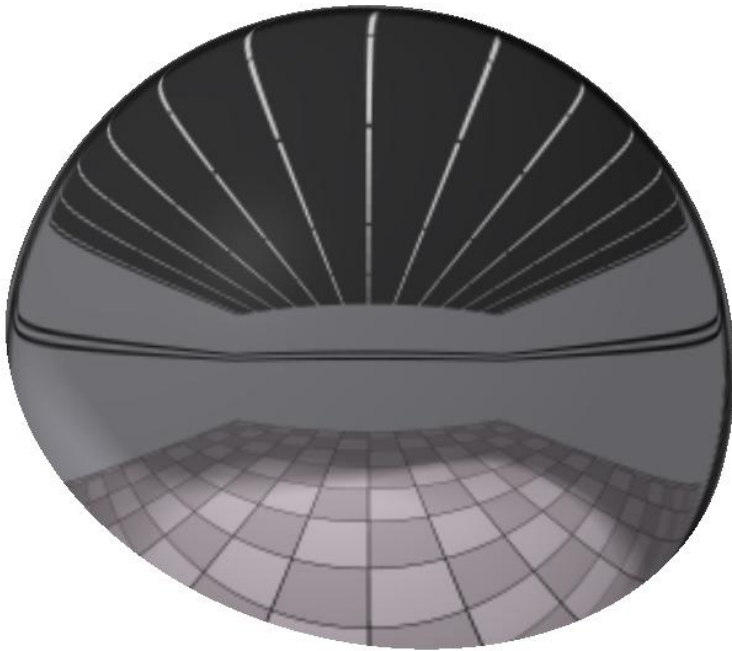
# Full-DG formulation extra benchmarks

- Neo-Hookean (elastic large deformations) plate ring loaded in a quasi-static way
  - The method gives accurate results even in the case of large distortions



## Full-DG formulation extra benchmarks

- $J_2$ -linear hardening (elasto-plastic large deformations) hemisphere loaded in a quasi-static way



# Full-DG formulation extra benchmarks

- $J_2$ -linear hardening (elasto-plastic large deformations) hemisphere loaded in a quasi-static way

– Same results for 2<sup>nd</sup> and 3<sup>rd</sup> order triangles

