Inequalities from two rows of a simplex tableau*

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Abstract. In this paper we explore the geometry of the integer points in a cone rooted at a rational point. This basic geometric object allows us to establish some links between lattice point free bodies and the derivation of inequalities for mixed integer linear programs by considering two rows of a simplex tableau simultaneously.

1 Introduction

Throughout this paper we investigate a mixed integer linear program (MIP) with rational data defined for a set I of integer variables and a set C of continuous variables

$$\text{(MIP)} \qquad \max\ c^T x \text{ subject to } Ax = b,\ x \geq 0,\ x_i \in \mathbb{Z} \text{ for } i \in I.$$

Let LP denote the linear programming relaxation of MIP. From the theory of linear programming it follows that a vertex x^* of the LP corresponds to a basic feasible solution of a simplex tableau associated with subsets B and N of basic and nonbasic variables

$$x_i + \sum_{j \in N} \bar{a}_{i,j} x_j = \bar{b}_i \text{ for } i \in B.$$

Any row associated with an index $i \in B \cap I$ such that $\bar{b}_i \notin \mathbb{Z}$ gives rise to a set

$$X(i) := \left\{ x \in \mathbb{R}^{|N|} \mid \bar{b}_i - \sum_{j \in N} \bar{a}_{i,j} x_j \in \mathbb{Z},\ x_j \geq 0 \text{ for all } j \in N \right\}$$

* This work was partly carried out within the framework of ADONET, a European network in Algorithmic Discrete Optimization, contract no. MRTN-CT-2003-504438. The second author is supported by FRS-FNRS. This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

whose analysis provides inequalities that are violated by x^*. Indeed, Gomory's mixed integer cuts [4] and mixed integer rounding cuts [6] are derived from such a basic set $X(i)$ using additional information about integrality of some of the variables. Interestingly, unlike in the pure integer case, no finite convergence proof of a cutting plane algorithm is known when Gomory's mixed integer cuts or mixed integer rounding cuts are applied only. More drastically, in [3], an interesting mixed integer program in three variables is presented, and it is shown that applying split cuts iteratively does not suffice to generate the cut that is needed to solve this problem.

Example 1: [3] Consider the mixed integer set

$$\begin{aligned} t &\leq x_1, \\ t &\leq x_2, \\ x_1 + x_2 + t &\leq 2, \\ x \in \mathbb{Z}^2 &\text{ and } t \in \mathbb{R}^1_+. \end{aligned}$$

The projection of this set onto the space of x_1 and x_2 variables is given by $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \leq 2\}$ and is illustrated in Fig. 1. A simple analysis shows that the inequality $x_1 + x_2 \leq 2$, or equivalently $t \leq 0$, is valid. In [3] it is, however, shown that with the objective function $z = \max t$, a cutting plane algorithm based on split cuts does not converge finitely. □

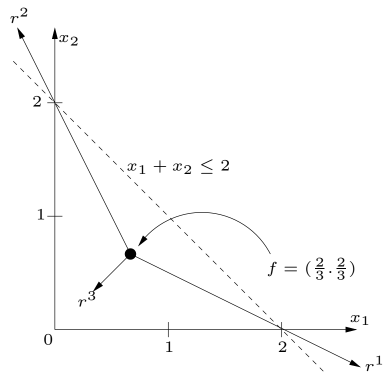


Fig. 1. The Instance in [3]

The analysis given in this paper will allow us to explain the cut $t \leq 0$ of Example 1. To this end we consider two indices $i_1, i_2 \in B \cap I$ simultaneously. It turns out that the underlying basic geometric object is significantly more complex than its one-variable counterpart. The set that we denote by $X(i_1, i_2)$

is described as

$$X(i_1,i_2) := \left\{x \in \mathbb{R}^{|N|} \mid \ \bar{b}_i - \sum_{j\in N} \bar{a}_{i,j}x_j \in \mathbb{Z} \text{ for } i = i_1, i_2, \ x_j \geq 0 \text{ for all } j \in N\right\}.$$

Setting

$$\begin{aligned} f &:= \left(\bar{b}_{i_1}, \bar{b}_{i_2}\right)^T \in \mathbb{R}^2, \text{ and} \\ r^j &:= \left(\bar{a}_{i_1,j}, \bar{a}_{i_2,j}\right)^T \in \mathbb{R}^2, \end{aligned}$$

the set obtained from two rows of a simplex tableau can be represented as

$$P_I := \{(x,s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j\in N} s_j r^j\},$$

where f is fractional and $r^j \in \mathbb{R}^2$ for all $j \in N$. Valid inequalities for the set P_I was studied in [5] by using superadditive functions related to the group problem associated with two rows. In this paper, we give a characterization of the facets of $\text{conv}(P_I)$ based on its geometry.

Example 1 (revisited): For the instance of Example 1, introduce slack variables, s_1, s_2 and y_1 in the three constraints. Then, solving as a linear program, the constraints of the optimal simplex tableau are

$$\begin{array}{lllllll} t & & & +\frac{1}{3}s_1 & +\frac{1}{3}s_2 & +\frac{1}{3}y_1 & = \frac{2}{3} \\ & x_1 & & -\frac{2}{3}s_1 & +\frac{1}{3}s_2 & +\frac{1}{3}y_1 & = \frac{2}{3} \\ & & x_2 & +\frac{1}{3}s_1 & -\frac{2}{3}s_2 & +\frac{1}{3}y_1 & = \frac{2}{3} \end{array}$$

Taking the last two rows, and rescaling using $s'_i = s_i/3$ for $i = 1, 2$, we obtain the set P_I

$$\begin{array}{lllllll} x_1 & & -2s'_1 & +1s'_2 & +\frac{1}{3}y_1 & = +\frac{2}{3} \\ & x_2 & +1s'_1 & -2s'_2 & +\frac{1}{3}y_1 & = +\frac{2}{3} \\ \multicolumn{6}{l}{x \in \mathbb{Z}^2, s \in \mathbb{R}^2_+, y_1 \in \mathbb{R}^1_+.} \end{array}$$

Letting $f = (\frac{2}{3}, \frac{2}{3})^T$, $r^1 = (2, -1)^T$, $r^2 = (-1, 2)^T$ and $r_3 = (-\frac{1}{3}, -\frac{1}{3})^T$ (see Fig. 1), one can derive a cut for $\text{conv}(P_I)$ of the form

$$x_1 + x_2 + y_1 \geq 2 \ \text{ or equivalently } t \leq 0,$$

which, when used in a cutting plane algorithm, yields immediate termination.

□

Our main contribution is to characterize geometrically all facets of $\text{conv}(P_I)$. All facets are *intersection cuts* [2], *i.e.*, they can be obtained from a (two-dimensional) convex body that does not contain any integer points in its interior. Our geometric approach is based on two important facts that we prove in this paper

– every facet is derivable from at most four nonbasic variables.

– with every facet F one can associate three or four particular vertices of $\text{conv}(P_I)$. The classification of F depends on how the corresponding $k = 3, 4$ integer points in $\mathbb{Z}^2$ can be partitioned into k sets of cardinality at most two.

More precisely, the facets of $\text{conv}(P_I)$ can be distinguished with respect to the number of sets that contain two integer points. Since $k = 3$ or $k = 4$, the following interesting situations occur

– no sets with cardinality two: all the $k \in \{3, 4\}$ sets contain exactly one tight integer point. We call cuts of this type *disection cuts*.
– exactly one set has cardinality two: in this case we show that the inequality can be derived from lifting a cut associated with a two-variable subproblem to k variables. We call these cuts *lifted two-variable cuts*.
– two sets have cardinality two. In this case we show that the corresponding cuts are *split cuts*.

Furthermore, we show that inequalities of the first two families are not split cuts. Our geometric approach allows us to generalize the cut introduced in Example 1. More specifically, the cut of Example 1 is a degenerate case in the sense that it is "almost" a disection cut and "almost" a lifted two-variable cut: by perturbing the vectors r^1, r^2 and r^3 slightly, the cut in Example 1 can become both a disection cut and a lifted two-variable cut.

We review some basic facts about the structure of $\text{conv}(P_I)$ in Section 2. In Section 3 we explore the geometry of all the feasible points that are tight for a given facet of $\text{conv}(P_I)$, explain our main result and presents the classification of all the facets of $\text{conv}(P_I)$.

2 Basic structure of $\text{conv}(P_I)$

The basic mixed-integer set considered in this paper is

$$P_I := \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} s_j r^j\}, \tag{1}$$

where $N := \{1, 2, \ldots, n\}$, $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $r^j \in \mathbb{Q}^2$ for all $j \in N$. The set $P_{LP} := \{(x, s) \in \mathbb{R}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} s_j r^j\}$ denotes the LP relaxation of P_I. The j^{th} unit vector in $\mathbb{R}^n$ is denoted e_j. In this section, we describe some basic properties of $\text{conv}(P_I)$. The vectors $\{r^j\}_{j \in N}$ are called *rays*, and we assume $r^j \neq 0$ for all $j \in N$.

In the remainder of the paper we assume $P_I \neq \emptyset$. The next lemma gives a characterization of $\text{conv}(P_I)$ in terms of vertices and extreme rays.

Lemma 1.

(i) The dimension of $\text{conv}(P_I)$ *is* n.
(ii) The extreme rays of $\text{conv}(P_I)$ *are* (r^j, e_j) *for* $j \in N$.
(iii) The vertices (x^I, s^I) *of* $\text{conv}(P_I)$ *take the following two forms:*

(a) $(x^I, s^I) = (x^I, s^I_j e_j)$*, where* $x^I = f + s^I_j r^j \in \mathbb{Z}^2$ *and* $j \in N$
(an integer point on the ray $\{f + s_j r^j : s_j \geq 0\}$*).*
(b) $(x^I, s^I) = (x^I, s^I_j e_j + s^I_k e_k)$*, where* $x^I = f + s^I_j r^j + s^I_k r^k \in \mathbb{Z}^2$ *and* $j, k \in N$
(an integer point in the set $f + \mathrm{cone}(\{r^j, r^k\})$*).*

Using Lemma 1, we now give a simple form for the valid inequalities for $\mathrm{conv}(P_I)$ considered in the remainder of the paper.

Corollary 1. *Every non-trivial valid inequality for* P_I *that is tight at a point* $(\bar{x}, \bar{s}) \in P_I$ *can be written in the form*

$$\sum_{j \in N} \alpha_j s_j \geq 1, \tag{2}$$

where $\alpha_j \geq 0$ *for all* $j \in N$.

For an inequality $\sum_{j \in N} \alpha_j s_j \geq 1$ of the form (2), let $N^0_\alpha := \{j \in N : \alpha_j = 0\}$ denote the variables with coefficient zero, and let $N^{\neq 0}_\alpha := N \setminus N^0_\alpha$ denote the remainder of the variables. We now introduce an object that is associated with the inequality $\sum_{j \in N} \alpha_j s_j \geq 1$. We will use this object to obtain a two dimensional representation of the facets of $\mathrm{conv}(P_I)$.

Lemma 2. *Let* $\sum_{j \in N} \alpha_j s_j \geq 1$ *be a valid inequality for* $\mathrm{conv}(P_I)$ *of the form (2). Define* $v^j := f + \frac{1}{\alpha_j} r^j$ *for* $j \in N^{\neq 0}_\alpha$*. Consider the convex polyhedron in* $\mathbb{R}^2$

$$L_\alpha := \{x \in \mathbb{R}^2 : \text{ there exists } s \in \mathbb{R}^n_+ \text{ s.t. } (x, s) \in P_{LP} \text{ and } \sum_{j \in N} \alpha_j s_j \leq 1\}.$$

(i) $L_\alpha = \mathrm{conv}(\{f\} \cup \{v^j\}_{j \in N^{\neq 0}_\alpha}) + \mathrm{cone}(\{r^j\}_{j \in N^0_\alpha})$.
(ii) $\mathrm{interior}(L_\alpha)$ *does not contain any integer points.*
(iii) If $\mathrm{cone}(\{r^j\}_{j \in N}) = \mathbb{R}^2$*, then* $f \in \mathrm{interior}(L_\alpha)$.

Example 2: Consider the set

$$P_I = \{(x, s) : x = f + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5\},$$

where $f = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$, and consider the inequality

$$2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7} s_5 \geq 1. \tag{3}$$

The corresponding set L_α is shown in Fig. 2. As can be seen from the figure, L_α does not contain any integer points in its interior. It follows that (3) is valid for $\mathrm{conv}(P_I)$. Note that, conversely, the coefficients α_j for $j = 1, 2, \ldots, 5$ can be obtained from the polygon L_α as follows: α_j is the ratio between the length of

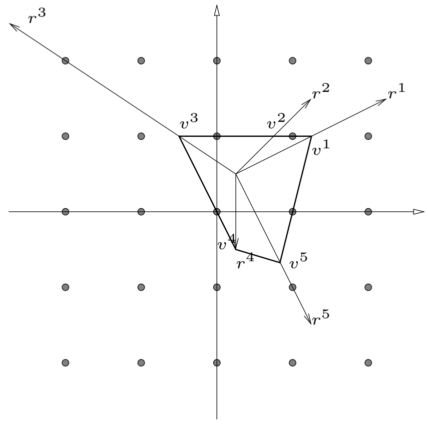


Fig. 2. The set L_α for a valid inequality for $\text{conv}(P_I)$

r^j and the distance between f and v^j. In particular, if the length of r^j is 1, then α_j is the inverse of the distance from f to v^j. □

The interior of L_α gives a two-dimensional representation of the points $x \in \mathbb{R}^2$ that are affected by the addition of the inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ to the LP relaxation P_{LP} of P_I. In other words, for any $(x, s) \in P_{LP}$ that satisfies $\sum_{j\in N} \alpha_j s_j < 1$, we have $x \in \text{interior}(L_\alpha)$. Furthermore, for a facet defining inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ of $\text{conv}(P_I)$, there exist n affinely independent points $(x^i, s^i) \in P_I$, $i = 1, 2, \ldots, n$, such that $\sum_{j\in N} \alpha_j s^i_j = 1$. The integer points $\{x^i\}_{i\in N}$ are on the boundary of L_α, *i.e.*, they belong to the integer set:

$$X_\alpha := \{x \in \mathbb{Z}^2 : \exists s \in \mathbb{R}^n_+ \text{ s.t. } (x, s) \in P_{LP} \text{ and } \sum_{j\in N} \alpha_j s_j = 1\}.$$

We have $X_\alpha = L_\alpha \cap \mathbb{Z}^2$, and $X_\alpha \neq \emptyset$ whenever $\sum_{j\in N} \alpha_j s_j \geq 1$ defines a facet of $\text{conv}(P_I)$. We first characterize the facets of $\text{conv}(P_I)$ that have zero coefficients.

Lemma 3. *Any facet defining inequality* $\sum_{j\in N} \alpha_j s_j \geq 1$ *for* $\text{conv}(P_I)$ *of the form (2) that has zero coefficients is a split cut. In other words, if* $N^0_\alpha \neq \emptyset$*, there exists* $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ *such that* $L_\alpha \subseteq \{(x_1, x_2) : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$.

Proof: Let $k \in N^0_\alpha$ be arbitrary. Then the line $\{f + \mu r^k : \mu \in \mathbb{R}\}$ does not contain any integer points. Furthermore, if $j \in N^0_\alpha$, $j \neq k$, is such that r^k and r^j are *not* parallel, then $f + \text{cone}(\{r^k, r^j\})$ contains integer points. It follows that all rays $\{r^j\}_{j\in N^0_\alpha}$ are parallel. By letting $\pi' := (r^k)^\perp = (-r^k_2, r^k_1)^T$ and $\pi'_0 := (\pi')^T f$, we have that $\{f+\mu r^k : \mu \in \mathbb{R}\} = \{x \in \mathbb{R}^2 : \pi'_1 x_1 + \pi'_2 x_2 = \pi'_0\}$. Now define:

$$\pi_0^1 := \max\{\pi'_1 x_1 + \pi'_2 x_2 : \pi'_1 x_1 + \pi'_2 x_2 \leq \pi'_0 \text{ and } x \in \mathbb{Z}^2\}, \text{ and}$$
$$\pi_0^2 := \min\{\pi'_1 x_1 + \pi'_2 x_2 : \pi'_1 x_1 + \pi'_2 x_2 \geq \pi'_0 \text{ and } x \in \mathbb{Z}^2\}.$$

We have $\pi_0^1 < \pi'_0 < \pi_0^2$, and the set $S_\pi := \{x \in \mathbb{R}^2 : \pi_0^1 \leq \pi'_1 x_1 + \pi'_2 x_2 \leq \pi_0^2\}$ does not contain any integer points in its interior. We now show $L_\alpha \subseteq S_\pi$ by showing that every vertex $v^m = f + \frac{1}{\alpha_m} r^m$ of L_α, where $m \in N_\alpha^{\neq 0}$, satisfies $v^m \in S_\pi$. Suppose v^m satisfies $\pi'_1 v_1^m + \pi'_2 v_2^m < \pi_0^1$ (the case $\pi'_1 v_1^m + \pi'_2 v_2^m > \pi_0^2$ is symmetric). By definition of π_0^1, there exists $x^I \in \mathbb{Z}^2$ such that $\pi'_1 x_1^I + \pi'_2 x_2^I = \pi_0^1$, and $x^I = \lambda v^m + (1-\lambda)(f + \delta r^k)$, where $\lambda \in]0,1[$, for some $\delta > 0$. We then have $x^I = f + \frac{\lambda}{\alpha_m} r^m + \delta(1-\lambda) r^k$. Inserting this representation of x^I into the inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ then gives $\alpha_m \frac{\lambda}{\alpha_m} + \alpha_k \delta(1-\lambda) = \lambda < 1$, which contradicts the validity of $\sum_{j \in N} \alpha_j s_j \geq 1$ for P_I. Hence $L_\alpha \subseteq S_\pi$.

To finish the proof, we show that we may write $S_\pi = \{x \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$ for some $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$. First observe that we can assume (by scaling) that π', π_0^1 and π_0^2 are integers. Next observe that any common divisor of π'_1 and π'_2 also divides both π_0^1 and π_0^2 (this follows from the fact that there exists $x^1, x^2 \in \mathbb{Z}^2$ such that $\pi'_1 x_1^1 + \pi'_2 x_2^1 = \pi_0^1$ and $\pi'_1 x_1^2 + \pi'_2 x_2^2 = \pi_0^2$). Hence we can assume that π'_1 and π'_2 are relative prime. Now the Integral Farkas Lemma (see [8]) implies that the set $\{x \in \mathbb{Z}^2 : \pi'_1 x_1 + \pi'_2 x_2 = 1\}$ is non-empty. It follows that we must have $\pi_0^2 = \pi_0^1 + 1$, since otherwise the point $\bar{y} := x' + x^1 \in \mathbb{Z}^2$, where $x' \in \{x \in \mathbb{Z}^2 : \pi'_1 x_1 + \pi'_2 x_2 = 1\}$ and $x^1 \in \{x \in \mathbb{Z}^2 : \pi'_1 x_1 + \pi'_2 x_2 = \pi_0^1\}$, satisfies $\pi_0^1 < \pi'_1 \bar{y}_1 + \pi'_2 \bar{y}_2 < \pi_0^2$, which contradicts that S_π does not contain any integer points in its interior. □

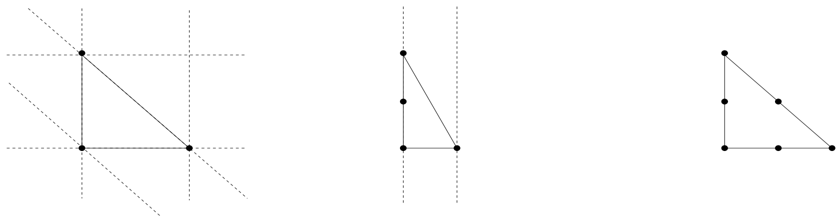
3 A characterization of conv(X_α) and conv(P_I)

As a preliminary step of our analysis, we first characterize the set $\text{conv}(X_\alpha)$. We assume $\alpha_j > 0$ for all $j \in N$. Clearly $\text{conv}(X_\alpha)$ is a convex polygon with only integer vertices, and since $X_\alpha \subseteq L_\alpha$, $\text{conv}(X_\alpha)$ does not have any integer points in its interior. We first limit the number of vertices of $\text{conv}(X_\alpha)$ to four (see [1] and [7] for this and related results).

Lemma 4. *Let $P \subset \mathbb{R}^2$ be a convex polygon with integer vertices that has no integer points in its interior.*

(i) P has at most four vertices
(ii) If P has four vertices, then at least two of its four facets are parallel.
(iii) If P is not a triangle with integer points in the interior of all three facets (see Fig. 3.(c)), then there exists parallel lines $\pi x = \pi_0$ and $\pi x = \pi_0 + 1$, $(\pi, \pi_0) \in \mathbb{Z}^3$, such that P is contained in the corresponding split set, i.e., $P \subseteq \{x \in \mathbb{R}^2 : \pi_0 \leq \pi x \leq \pi_0 + 1\}$.

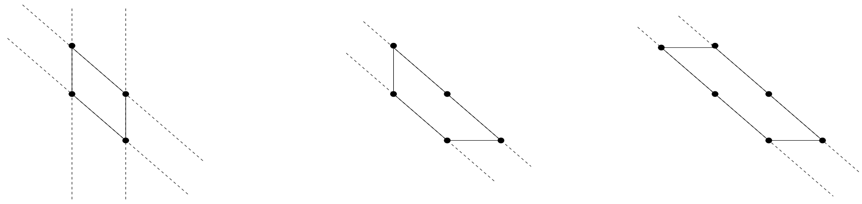
Lemma 4 shows that the polygons in Fig. 3 include all possible polygons that can be included in the set L_α in the case when L_α is bounded and of dimension 2. The dashed lines in Fig. 3 indicate the possible split sets that include P. We excluded from Fig. 3 the cases when X_α is of dimension 1. We note that Lemma 4.(iii) (existence of split sets) proves that there cannot be any triangles where two facets have interior integer points, and also that no quadrilateral can have more than two facets that have integer points in the interior.



(a) A triangle: no facet has interior integer points

(b) A triangle: one facet has interior integer points

(c) A triangle: all facets have interior integer points



(d) A quadrilateral: no facet has interior integer points

(e) A quadrilateral: one facet has interior integer points

(f) A quadrilateral: two facets have interior integer points

Fig. 3. All integer polygons that do not have interior integer points

Next, we focus on the set L_α. As before we assume $\alpha_j > 0$ for all $j \in N$. Due to the direct correspondence between the set L_α and a facet defining inequality $\sum_{j\in N} \alpha_j s_j \geq 1$ for $\text{conv}(P_I)$, this gives a characterization of the facets of $\text{conv}(P_I)$. The main result in this section is that L_α can have at most four vertices. In other words, we prove

Theorem 1. *Let* $\sum_{j\in N} \alpha_j s_j \geq 1$ *be a facet defining inequality for* $\text{conv}(P_I)$ *that satisfies* $\alpha_j > 0$ *for all* $j \in N$. *Then* L_α *is a polygon with at most four vertices.*

Theorem 1 shows that there exists a set $S \subseteq N$ such that $|S| \leq 4$ and $\sum_{j\in S} \alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I(S))$, where

$$P_I(S) := \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}_+^{|S|} : x = f + \sum_{j\in S} s_j r^j\}.$$

Throughout this section we assume that no two rays point in the same direction. If two variables $j_1, j_2 \in N$ are such that $j_1 \neq j_2$ and $r^{j_1} = \delta r^{j_2}$ for some $\delta > 0$, then the halflines $\{x \in \mathbb{R}^2 : x = f + s_{j_1} r^{j_1}, s_{j_1} \geq 0\}$ and $\{x \in \mathbb{R}^2 : x = f + s_{j_2} r^{j_2}, s_{j_2} \geq 0\}$ intersect the boundary of L_α at the same point, and therefore $L_\alpha = \text{conv}(\{f\} \cup \{v^j\}_{j \in N}) = \text{conv}(\{f\} \cup \{v^j\}_{j \in N \setminus \{j_2\}})$, where $v^j := f + \frac{1}{\alpha_j} r^j$ for $j \in N$. This assumption does therefore not affect the validity of Theorem 1.

The proof of Theorem 1 is based on characterizing the vertices $\text{conv}(P_I)$ that are tight for $\sum_{j \in N} \alpha_j s_j \geq 1$. We show that there exists a subset $S \subseteq N$ of variables and a set of $|S|$ affinely independent vertices of $\text{conv}(P_I)$ such that $|S| \leq 4$ and $\{\alpha_j\}_{j \in S}$ is the unique solution to the equality system of the polar defined by these vertices. The following notation will be used intensively in the remainder of this section.

Notation 1

(i) The number $k \leq 4$ denotes the number of vertices of $\text{conv}(X_\alpha)$.
(ii) The set $\{x^v\}_{v \in K}$ denotes the vertices of $\text{conv}(X_\alpha)$, *where* $K := \{1, 2, \ldots, k\}$.

Recall that Lemma 1.(iii) demonstrates that for a vertex $(\bar{x}, \bar{s})$ of $\text{conv}(P_I)$, $\bar{s}$ is positive on at most two coordinates $j_1, j_2 \in N$, and in that case $\bar{x} \in f + \text{cone}(\{r^{j_1}, r^{j_2}\})$. If $\bar{s}$ is positive on only one coordinate $j \in N$, then $\bar{x} = f + \bar{s}_j r^j$, and the inequality of the polar corresponding to $(\bar{x}, \bar{s})$ is $\alpha_j \bar{s}_j \geq 1$, which simply states $\alpha_j \geq \frac{1}{\bar{s}_j}$. A point $\bar{x} \in \mathbb{Z}^2$ that satisfies $\bar{x} \in \{x \in \mathbb{R}^2 : x = f + s_j r^j, s_j \geq 0\}$ for some $j \in N$ is called a *ray point* in the remainder of the paper. In order to characterize the tight inequalities of the polar that correspond to vertices x^v of $\text{conv}(X_\alpha)$ that are *not* ray points, we introduce the following concepts.

Definition 1. *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be valid for* $\text{conv}(P_I)$. *Suppose $\bar{x} \in \mathbb{Z}^2$ is* not *a ray point, and that $\bar{x} \in f +$* $\text{cone}(\{r^{j_1}, r^{j_2}\})$, *where $j_1, j_2 \in N$. This implies $\bar{x} = f + s_{j_1} r^{j_1} + s_{j_2} r^{j_2}$, where $s_{j_1}, s_{j_1} > 0$ are unique.*

(a) The pair (j_1, j_2) is said to give a representation *of $\bar{x}$.*
(b) If $\alpha_{j_1} s_{j_1} + \alpha_{j_2} s_{j_2} = 1$, (j_1, j_2) is said to give a tight *representation of $\bar{x}$ wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$.*
(c) If $(i_1, i_2) \in N \times N$ satisfies $\text{cone}(\{r^{i_1}, r^{i_2}\}) \subseteq \text{cone}(\{r^{j_1}, r^{j_2}\})$, *the pair (i_1, i_2) is said to define a* subcone *of (j_1, j_2).*

Example 2 (continued): Consider again the set

$$P_I = \{(x, s) : x = f + \begin{pmatrix} 2 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -3 \\ 2 \end{pmatrix} s_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_4 + \begin{pmatrix} 1 \\ -2 \end{pmatrix} s_5\},$$

where $f = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$, and the valid inequality $2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7} s_5 \geq 1$ for $\text{conv}(P_I)$. The point $\bar{x} = (1, 1)$ is on the boundary of L_α (see Fig. 2). We have that $\bar{x}$ can be written in any of the following forms

$$\begin{aligned}
\bar{x} &= f + \frac{1}{4}r^1 + \frac{1}{4}r^2, \\
\bar{x} &= f + \frac{3}{7}r^1 \qquad + \frac{1}{28}r^3, \\
\bar{x} &= f \qquad + \frac{3}{4}r^2 \qquad + \frac{1}{4}r^4.
\end{aligned}$$

It follows that $(1,2), (1,3)$ and $(2,4)$ all give representations of $\bar{x}$. Note that $(1,2)$ and $(1,3)$ give tight representations of $\bar{x}$ wrt. the inequality $2s_1 + 2s_2 + 4s_3 + s_4 + \frac{12}{7}s_5 \geq 1$, whereas $(2,4)$ does not. Finally note that $(1,5)$ defines a subcone of $(2,4)$. □

Observe that, for a vertex x^v of $\text{conv}(X_\alpha)$ which is not a ray point, and a tight representation (j_1, j_2) of x^v, the corresponding inequality of the polar satisfies $\alpha_{j_1} t_{j_1} + \alpha_{j_2} t_{j_2} = 1$, where $t_{j_1}, t_{j_2} > 0$. We now characterize the set of tight representations of an integer point $\bar{x} \in \mathbb{Z}^2$, which is not a ray point

$$T_\alpha(\bar{x}) := \{(j_1, j_2) : (j_1, j_2) \text{ gives a tight representation of } \bar{x} \text{ wrt. } \sum_{j \in N} \alpha_j s_j \geq 1\}.$$

We show that $T_\alpha(\bar{x})$ contains a unique maximal representation $(j_1^{\bar{x}}, j_2^{\bar{x}}) \in T_\alpha(\bar{x})$ with the following properties.

Lemma 5. *There exists a unique maximal representation $(j_1^{\bar{x}}, j_2^{\bar{x}}) \in T_\alpha(\bar{x})$ of $\bar{x}$ that satisfies:*

(i) Every subcone (j_1, j_2) of $(j_1^{\bar{x}}, j_2^{\bar{x}})$ that gives a representation of $\bar{x}$ satisfies $(j_1, j_2) \in T_\alpha(\bar{x})$.
(ii) Conversely, every $(j_1, j_2) \in T_\alpha(\bar{x})$ defines a subcone of $(j_1^{\bar{x}}, j_2^{\bar{x}})$.

To prove Lemma 5, there are two cases to consider. For two representations (i_1, i_2) and (j_1, j_2) of $\bar{x}$, either one of the two cones (i_1, i_2) and (j_1, j_2) is contained in the other (Lemma 6), or their intersection defines a subcone of both (i_1, i_2) and (j_1, j_2) (Lemma 7). Note that we cannot have that their intersection is empty, since they both give a representation of $\bar{x}$.

Lemma 6. *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\text{conv}(P_I)$ that satisfies $\alpha_j > 0$ for all $j \in N$, and let $\bar{x} \in \mathbb{Z}^2$. Then $(j_1, j_2) \in T_\alpha(\bar{x})$ implies $(i_1, i_2) \in T_\alpha(\bar{x})$ for every subcone (i_1, i_2) of (j_1, j_2) that gives a representation of $\bar{x}$.*

Proof: Suppose $(j_1, j_2) \in T_\alpha(\bar{x})$. Observe that it suffices to prove the following: for any $j_3 \in N$ such that $r^{j_3} \in \text{cone}(\{r^{j_1}, r^{j_2}\})$ and (j_1, j_3) gives a representation of $\bar{x}$, the representation (j_1, j_3) is tight wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$. The result for all remaining subcones of (j_1, j_2) follows from repeated application of this result. For simplicity we assume $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$.

Since $\bar{x} \in f + \text{cone}(\{r^1, r^2\})$, $\bar{x} \in f + \text{cone}(\{r^1, r^3\})$ and $r^3 \in \text{cone}(\{r^1, r^2\})$, we may write $\bar{x} = f + u_1 r^1 + u_2 r^2$, $\bar{x} = f + v_1 r^1 + v_3 r^3$ and $r^3 = w_1 r^1 + w_2 r^2$, where $u_1, u_2, v_1, v_3, w_1, w_2 \geq 0$. Furthermore, since $(1,2)$ gives a tight representation of $\bar{x}$ wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$, we have $\alpha_1 u_1 + \alpha_2 u_2 = 1$. Finally we have $\alpha_1 v_1 + \alpha_3 v_3 \geq 1$, since $\sum_{j \in N} \alpha_j s_j \geq 1$ is valid for P_I. If also $\alpha_1 v_1 + \alpha_3 v_3 = 1$, we are done, so suppose $\alpha_1 v_1 + \alpha_3 v_3 > 1$.

We first argue that this implies $\alpha_3 > \alpha_1 w_1 + \alpha_2 w_2$. Since $\bar{x} = f + u_1 r^1 + u_2 r^2 = f + v_1 r^1 + v_3 r^3$, it follows that $(u_1 - v_1) r^1 = v_3 r^3 - u_2 r^2$. Now, using the representation $r^3 = w_1 r^1 + w_2 r^2$, we get $(u_1 - v_1 - v_3 w_1) r^1 + (u_2 - v_3 w_2) r^2 = 0$. Since r^1 and r^2 are linearly independent, we obtain:

$$(u_1 - v_1) = v_3 w_1 \text{ and } u_2 = v_3 w_2.$$

Now we have $\alpha_1 v_1 + \alpha_3 v_3 > 1 = \alpha_1 u_1 + \alpha_2 u_2$, which implies $(v_1 - u_1)\alpha_1 - \alpha_2 u_2 + \alpha_3 v_3 > 0$. Using the identities derived above, we get $-v_3 w_1 \alpha_1 - \alpha_2 v_3 w_2 + \alpha_3 v_3 > 0$, or equivalently $v_3(-w_1 \alpha_1 - \alpha_2 w_2 + \alpha_3) > 0$. It follows that $\alpha_3 > \alpha_1 w_1 + \alpha_2 w_2$.

We now derive a contradiction to the identity $\alpha_3 > \alpha_1 w_1 + \alpha_2 w_2$. Since $\sum_{j \in N} \alpha_j s_j \geq 1$ defines a facet of $\text{conv}(P_I)$, there must exist $x' \in \mathbb{Z}^2$ and $k \in N$ such that $(3, k)$ gives a tight representation of x' wrt. $\sum_{j \in N} \alpha_j s_j \geq 1$. In other words, there exists $x' \in \mathbb{Z}^2$, $k \in N$ and $\delta_3, \delta_k \geq 0$ such that $x' = f + \delta_3 r^3 + \delta_k r^k$ and $\alpha_3 \delta_3 + \alpha_k \delta_k = 1$. Furthermore, we can choose x', δ_3 and δ_k such that r^3 is used in the representation of x', *i.e.*, we can assume $\delta_3 > 0$.

Now, using the representation $r^3 = w_1 r^1 + w_2 r^2$ then gives $x' = f + \delta_3 r^3 + \delta_k r^k = f + \delta_3 w_1 r^1 + \delta_3 w_2 r^2 + \delta_k r^k$. Since $\sum_{j \in N} \alpha_j s_j \geq 1$ is valid for P_I, we have $\alpha_1 \delta_3 w_1 + \alpha_2 \delta_3 w_2 + \alpha_k \delta_k \geq 1 = \alpha_3 \delta_3 + \alpha_k \delta_k$. This implies $\delta_3(\alpha_3 - \alpha_1 w_1 - \alpha_2 w_2) \leq 0$, and therefore $\alpha_3 \leq \alpha_1 w_1 - \alpha_2 w_2$, which is a contradiction. □

Lemma 7. *Let* $\sum_{j \in N} \alpha_j s_j \geq 1$ *be a facet defining inequality for* $\text{conv}(P_I)$ *satisfying* $\alpha_j > 0$ *for* $j \in N$*, and suppose* $\bar{x} \in \mathbb{Z}^2$ *is* not *a ray point. Also suppose the intersection between the cones* $(j_1, j_2), (j_3, j_4) \in T_\alpha(\bar{x})$ *is given by the subcone* (j_2, j_3) *of both* (j_1, j_2) *and* (j_3, j_4)*. Then* $(j_1, j_4) \in T_\alpha(\bar{x})$*, i.e.,* (j_1, j_4) *also gives a tight representation of* $\bar{x}$*.*

Proof: For simplicity assume $j_1 = 1$, $j_2 = 2$, $j_3 = 3$ and $j_4 = 4$. Since the cones $(1,2)$ and $(3,4)$ intersect in the subcone $(2,3)$, we have $r^3 \in \text{cone}(\{r^1, r^2\})$, $r^2 \in \text{cone}(\{r^3, r^4\})$, $r^4 \notin \text{cone}(\{r^1, r^2\})$ and $r^1 \notin \text{cone}(\{r^3, r^4\})$. We first represent $\bar{x}$ in the translated cones in which we have a tight representation of $\bar{x}$. In other words, we can write

$$\bar{x} = f + u_1 r^1 + u_2 r^2, \quad (4)$$

$$\bar{x} = f + v_3 r^3 + v_4 r^4 \text{ and} \quad (5)$$

$$\bar{x} = f + z_2 r^2 + z_3 r^3, \quad (6)$$

where $u_1, u_2, v_3, v_4, z_2, z_3 > 0$. Note that Lemma 6 proves that (6) gives a tight represention of $\bar{x}$. Using (4)-(6), we obtain the relation

$$\binom{T_{1,1}I_2\ T_{1,2}I_2}{T_{2,1}I_2\ T_{2,2}I_2}\binom{r^2}{r^3} = \binom{u_1 r^1}{v_4 r^4}, \tag{7}$$

where T is the 2×2 matrix $T := \binom{T_{1,1}\ T_{1,2}}{T_{2,1}\ T_{2,2}} = \begin{pmatrix}(z_2 - u_2) & z_3 \\ z_2 & (z_3 - v_3)\end{pmatrix}$ and I_2 is the 2×2 identity matrix. On the other hand, the tightness of the representations (4)-(6) leads to the following identities

$$\alpha_1 u_1 + \alpha_2 u_2 = 1, \tag{8}$$

$$\alpha_3 v_3 + \alpha_4 v_4 = 1 \text{ and} \tag{9}$$

$$\alpha_2 z_2 + \alpha_4 z_3 = 1, \tag{10}$$

where, again, the last identity follows from Lemma 6. Using (8)-(10), we obtain the relation

$$\binom{T_{1,1}\ T_{1,2}}{T_{2,1}\ T_{2,2}}\binom{\alpha_2}{\alpha_3} = \binom{u_1\alpha_1}{v_4\alpha_4}. \tag{11}$$

We now argue that T is non-singular. Suppose, for a contradiction, that $T_{1,1}T_{2,2} = T_{1,2}T_{2,1}$. From (5) and (6) we obtain $v_4 r^4 = (z_3 - v_3)r^3 + z_2 r^2$, which implies $z_3 < v_3$, since $r^4 \notin \text{cone}(\{r^1, r^2\}) \supseteq \text{cone}(\{r^2, r^3\})$. Multiplying the first equation of (11) with $T_{2,2}$ gives $T_{2,2}T_{1,1}\alpha_2 + T_{2,2}T_{1,2}\alpha_3 = u_1 T_{2,2}\alpha_1$, which implies $T_{1,2}(T_{2,1}\alpha_2 + T_{2,2}\alpha_3) = u_1 T_{2,2}\alpha_1$. By using the definition of T, this can be rewritten as $z_3(\alpha_2 z_2 + (z_3 - v_3)\alpha_3) = u_1\alpha_1(z_3 - v_3)$. Since $z_2\alpha_2 + z_3\alpha_3 = 1$, this implies $z_3(1 - v_3\alpha_3) = u_1\alpha_1(z_3 - v_3)$. However, from (9) we have $v_3\alpha_3 \in]0, 1[$, so $z_3(1 - v_3\alpha_3) > 0$ and $u_1\alpha_1(z_3 - v_3) < 0$, which is a contradiction. Hence T is non-singular.

We now solve (7) for an expression of r^2 and r^3 in terms of r^1 and r^4. The inverse of the coefficient matrix on the left hand side of (7) is given by $\binom{T^{-1}_{1,1}I_2\ T^{-1}_{1,2}I_2}{T^{-1}_{2,1}I_2\ T^{-1}_{2,2}I_2}$, where $T^{-1} := \binom{T^{-1}_{1,1}\ T^{-1}_{1,2}}{T^{-1}_{2,1}\ T^{-1}_{2,2}}$ denotes the inverse of T. We therefore obtain

$$r^2 = \lambda_1 r^1 + \lambda_4 r^4 \ \text{ and} \tag{12}$$

$$r^3 = \mu_1 r^1 + \mu_4 r^4, \tag{13}$$

where $\lambda_1 := u_1 T^{-1}_{1,1}, \lambda_4 := v_4 T^{-1}_{1,2}$, $\mu_1 := u_1 T^{-1}_{2,1}$ and $\mu_4 := v_4 T^{-1}_{2,2}$. Similarly, solving (11) to express α_2 and α_3 in terms of α_1 and α_4 gives

$$\alpha_2 = \lambda_1\alpha_1 + \lambda_4\alpha_4 \ \text{ and} \tag{14}$$

$$\alpha_3 = \mu_1\alpha_1 + \mu_4\alpha_4. \tag{15}$$

Now, using for instance (4) and (12), we obtain

$$\bar{x} = f + (u_1 + u_2\lambda_1)r^1 + (u_2\lambda_4)r^4, \quad \text{and:}$$

$$\begin{array}{ll} (u_1 + u_2\lambda_1)\alpha_1 + (u_2\lambda_4)\alpha_4 & = \quad \text{(using (8))} \\ (1 - u_2\alpha_2) + u_2\lambda_1\alpha_1 + (u_2\lambda_4)\alpha_4 & = \\ 1 + u_2(\lambda_1\alpha_1 + \lambda_4\alpha_4 - \alpha_2) & = 1. \text{ (using (14))} \end{array}$$

To finish the proof, we only need to argue that we indeed have $\bar{x} \in f + \text{cone}(\{r^1, r^4\})$, *i.e.*, that $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4$ with $\delta_1 = u_1 + u_2\lambda_1$ and $\delta_4 = u_2\lambda_4$ satisfying $\delta_1, \delta_4 \geq 0$. If $\delta_1 \leq 0$ and $\delta_4 > 0$, we have $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4 = f + u_1 r^1 + u_2 r^2$, which means $\delta_4 r^4 = (u_1 - \delta_1)r^1 + u_2 r^2 \in \text{cone}(\{r^1, r^2\})$, which is a contradiction. Similarly, if $\delta_1 > 0$ and $\delta_4 \leq 0$, we have $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4 = f + v_3 r^3 + v^4 r^4$, which implies $\delta_1 r^1 = v_3 r^3 + (v_4 - \delta_4) r^4 \in \text{cone}(\{r^3, r^4\})$, which is also a contradiction. Hence we can assume $\delta_1, \delta_4 \leq 0$. However, since $\delta_1 = u_1 + u_2\lambda_1$ and $\delta_4 = u_2\lambda_4$, this implies $\lambda_1, \lambda_4 \leq 0$, and this contradicts what was shown above, namely that the representation $\bar{x} = f + \delta_1 r^1 + \delta_4 r^4$ satisfies $\alpha_1\delta_1 + \alpha_4\delta_4 = 1$. □

It follows that only one tight representation of every point x of $\text{conv}(X_\alpha)$ is needed. We now use Lemma 5 to limit the number of vertices of L_α to four. The following notation is introduced. The set $J^x := \cup_{(j_1,j_2) \in T_\alpha(x)} \{j_1, j_2\}$ denotes the set of variables that are involved in tight representations of x. As above, $(j_1^x, j_2^x) \in T_\alpha(x)$ denotes the unique maximal representation of x. Furthermore, given any $(j_1, j_2) \in T_\alpha(x)$, let $(t_{j_1}^{j_2}(x), t_{j_2}^{j_1}(x))$ satisfy $x = f + t_{j_1}^{j_2}(x) r^{j_1} + t_{j_2}^{j_1}(x) r^{j_2}$. Lemma 5 implies that $r^j \in \text{cone}(r^{j_1^x}, r^{j_2^x})$ for every $j \in J^x$. Let $(w_1^j(x), w_2^j(x))$ satisfy $r^j = w_1^j(x) r^{j_1^x} + w_2^j(x) r^{j_2^x}$, where $w_1^j(x), w_2^j(x) \geq 0$ are unique.

Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a valid inequality for $\text{conv}(P_I)$ that satisfies $\alpha_j > 0$ for $j \in N$. The inequality $\sum_{j \in N} \alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I)$, if and only if the coefficients $\{\alpha_j\}_{j \in N}$ define a vertex of the polar of $\text{conv}(P_I)$. Hence $\sum_{j \in N} \alpha_j s_j \geq 1$ is facet defining for $\text{conv}(P_I)$, if and only if the solution to the system

$$\alpha_{j_1} t_{j_1}^{j_2}(x) + \alpha_{j_2} t_{j_2}^{j_1}(x) = 1, \quad \text{for every } x \in X_\alpha \text{ and } (j_1, j_2) \in T_\alpha(x). \tag{16}$$

is unique. We now rewrite the subsystem of (16) that corresponds to a fixed point $x \in X_\alpha$.

Lemma 8. *Let $\sum_{j \in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\text{conv}(P_I)$ that satisfies $\alpha_j > 0$ for $j \in N$. Suppose $x \in X_\alpha$ is not a ray point. The system*

$$\alpha_{j_1} t_{j_1}^{j_2}(x) + \alpha_{j_2} t_{j_2}^{j_1}(x) = 1, \quad \textit{for every } (j_1, j_2) \in T_\alpha(x). \tag{17}$$

has the same set of solutions $\{\alpha_j\}_{j \in J^x}$ as the system

$$1 = t_{j_1}^{j_2}(x)\alpha_{j_1} + t_{j_2}^{j_1}(x)\alpha_{j_2}, \qquad \textit{for } (j_1, j_2) = (j_1^x, j_2^x), \tag{18}$$

$$\alpha_j = w_1^j(x)\alpha_{j_1^x} + w_2^j(x)\alpha_{j_2^x}, \qquad \textit{for } j \in J^x \setminus \{j_1^x, j_2^x\}. \tag{19}$$

We next show that it suffices to consider vertices of $\mathrm{conv}(X_\alpha)$ in (16).

Lemma 9. *Let $\sum_{j\in N} \alpha_j s_j \geq 1$ be a facet defining inequality for $\mathrm{conv}(P_I)$ that satisfies $\alpha_j > 0$ for $j \in N$. Suppose $x \in X_\alpha$ is not a vertex of $\mathrm{conv}(X_\alpha)$. Then there exists vertices y and z of $\mathrm{conv}(X_\alpha)$ such that the equalities*

$$\alpha_{j_1} t_{j_1}^{j_2}(y) + \alpha_{j_2} t_{j_2}^{j_1}(y) = 1, \quad \text{for every } (j_1, j_2) \in T_\alpha(y) \text{ and} \tag{20}$$

$$\alpha_{j_1} t_{j_1}^{j_2}(z) + \alpha_{j_2} t_{j_2}^{j_1}(z) = 1, \quad \text{for every } (j_1, j_2) \in T_\alpha(z) \tag{21}$$

imply the equalities:

$$\alpha_{j_1} t_{j_1}^{j_2}(x) + \alpha_{j_2} t_{j_2}^{j_1}(x) = 1, \quad \text{for every } (j_1, j_2) \in T_\alpha(x). \tag{22}$$

By combining Lemma 8 and Lemma 9 we have that, if the solution to (16) is unique, then the solution to the system

$$t_{j_1^x}^{j_2^x}(x)\alpha_{j_1^x} + t_{j_2^x}^{j_1^x}(x)\alpha_{j_2^x} = 1, \qquad \text{for every vertex } x \text{ of } \mathrm{conv}(X_\alpha). \tag{23}$$

is unique. Since (23) involves exactly $k \leq 4$ equalities and has a unique solution, exactly $k \leq 4$ variables are involved in (23) as well. This finishes the proof of Theorem 1.

We note that from an inequality $\sum_{j\in S} \alpha_j s_j \geq 1$ that defines a facet of $\mathrm{conv}(P_I(S))$, where $|S| = k$, the coefficients on the variables $j \in N \setminus S$ can be simultaneously lifted by computing the intersection point between the halfline $\{f + s_j r^j : s_j \geq 0\}$ and the boundary of L_α.

We now use Theorem 2 to categorize the inequalities $\sum_{j\in N} \alpha_j s_j \geq 1$ that define facets of $\mathrm{conv}(P_I)$. For simplicity, we only consider the most general case, namely when none of the vertices of $\mathrm{conv}(X_\alpha)$ are ray points. Furthermore, we only consider $k = 3$ and $k = 4$. When $k = 2$, $\sum_{j\in N} \alpha_j s_j \geq 1$ is a facet defining inequality for a cone defined by two rays. We divide the remaining facets of $\mathrm{conv}(P_I)$ into the following three main categories.

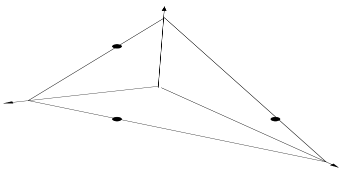
(i) *Disection cuts (Fig. 4.(a) and Fig. 4.(b)):*
Every vertex of $\mathrm{conv}(X_\alpha)$ belongs to a different facet of L_α.

(ii) *Lifted two-variable cuts (Fig. 4.(c) and Fig. 4.(d)):*
Exactly one facet of L_α contains two vertices of $\mathrm{conv}(X_\alpha)$. Observe that this implies that there is a set $S \subset N$, $|S| = 2$, such that $\sum_{j\in S} \alpha_j s_j \geq 1$ is facet defining for $\mathrm{conv}(P_I(S))$.

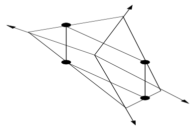
(iii) *Split cuts:*
Two facets of L_α each contain two vertices of $\mathrm{conv}(X_\alpha)$.

An example of a cut that is not a split cut was given in [3] (see Fig. 1). This cut is the only cut when $\mathrm{conv}(X_\alpha)$ is the triangle of Fig. 4.(c), and, necessarily, $L_\alpha = \mathrm{conv}(X_\alpha)$ in this case. Hence, *all* three rays that define this triangle are ray points. As mentioned in the introduction, the cut in [3] can be viewed as being on the boundary between disection cuts and lifted two-variable cuts.

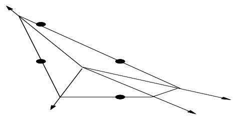
Since the cut presented in [3] is not a split cut, and this cut can be viewed as being on the boundary between disection cuts and lifted two-variable cuts, a natural question is whether or not disection cuts and lifted two-variable cuts are split cuts. We finish this section by answering this question.



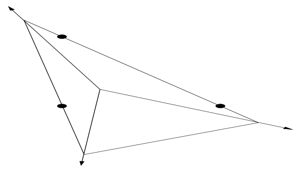
(a) Disection cut from a triangle



(b) Disection cut from a quadrilateral



(c) Lifted two-variable cut from quadrilateral



(d) Lifted two-variable cut from triangle

Fig. 4. Disection cuts and lifted two-variable cuts

Lemma 10. *Let* $\sum_{j \in N} \alpha_j s_j \geq 1$ *be a facet defining inequality for* $\text{conv}(P_I)$ *satisfying* $\alpha_j > 0$ *for* $j \in N$. *Also suppose* $\sum_{j \in N} \alpha_j s_j \geq 1$ *is either a disection cut or a lifted two-variable cut. Then* $\sum_{j \in N} \alpha_j s_j \geq 1$ *is* not *a split cut.*

Proof: Observe that, if $\sum_{j \in N} \alpha_j s_j \geq 1$ is a split cut, then there exists $(\pi, \pi_0) \in \mathbb{Z}^2 \times \mathbb{Z}$ such that L_α is contained in the split set $S_\pi := \{x \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$. Furthermore, all points $x \in X_\alpha$ and all vertices of L_α must be either on the line $\pi^T x = \pi_0$, or on the line $\pi^T x = \pi_0 + 1$. However, this implies that there must be two facets of L_α that do not contain any integer points. □

References

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