Algebraic approach to modal extensions of Łukasiewicz logics

Doctoral dissertation

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Résumé

Nous consacrons cette dissertation à une étude algébrique de certaines généralisations multivaluées des logiques modales. Notre point de départ est la définition des modèle de Kripke $[0,1]$-valués et $Ł_n$-valués, où $[0,1]$ désigne la MV-algèbre bien connue et $Ł_n$ sa sous-algèbre $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ pour tout naturel non nul n.

Nous utilisons deux types de structures pour définir une relation de validité : la classe des $\mathcal{L}$-structures et celles des $\mathcal{L}$-structures $Ł_n$-valuées. Ces dernières sont des $\mathcal{L}$-structures dans lesquelles nous précisons pour chaque monde u l'ensemble $Ł_m$ (où m est un diviseur de n) des valeurs de vérité que les formules sont autorisées à prendre en u.

Ces deux classes de structures définissent deux notions distinctes de validité. Nous les utilisons pour étudier le problème de la définissabilité des classes de structures à l'aide du langage modal. Nous obtenons dans les deux cas l'équivalent du théorème de Goldblatt - Thomason.

Nous considérons aussi les problèmes de complétude vis à vis de ces sémantiques relationnelles à l'aide des liens qui les lient à la sémantique algébrique. Les résultats les plus forts que nous obtenons concernent les logiques modales $Ł_n$-valuées. En effet, dans ce cas, nous pouvons appliquer et développer des outils algébriques (à savoir, les extensions canoniques et les extensions canoniques fortes) qui permettent de générer des logiques complètes.

Abstract

This dissertation is focused on an algebraic approach of some many-valued generalizations of modal logics. The starting point is the definition of the $[0,1]$-valued and the $Ł_n$-valued Kripke models, where $[0,1]$ denotes the well known MV-algebra and $Ł_n$ its finite subalgebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ for any positive integer n.

Two types of structures are used to define validity of formulas: the class of $\mathcal{L}$-frames and the class of $Ł_n$-valued $\mathcal{L}$-frames. The latter structures are $\mathcal{L}$-frames in which we specify in each world u the set $Ł_m$ (where m is a divisor of n) of the possible truth values of the formulas in u.

These two classes of structures define two distinct notions of validity. We use these notions to study the problem of definability of classes of structures with modal formulas. We obtain for these two classes an equivalent of the Goldblatt - Thomason theorem.

We are able to consider completeness problems with respect to these relational semantics thanks to the connections between relational and algebraic semantics. Our strongest results are about $Ł_n$-valued logic. We are indeed able to apply and develop algebraic tools (namely, canonical and strong canonical extensions) that allow to generate complete $Ł_n$-valued logics.

Thanks

I would like to thank Georges HANSOUL who led me to the problem of the algebraic approach of many-valued modal logics and who helped me when I was facing difficulties by providing me with, sometimes advices, sometimes answers.

I also thank my parents for the support they give me and my wife in everyday life and during the most painful moments.

Since the publication of this dissertation coincide more or less with the end of my appointment as an Assistant at the Department of Mathematic of the University of Liège, I would like to express my gratitude to my colleagues with whom I had very good times.

But above all, I would like to thank my wife Sarah for proving everyday to her family and relatives that it is possible and even necessary in all circumstances to live happier each day than the day before. Thank you, Sarah, for the way you lead your life, guide mine and for your constant struggle to live. It is really the greatest proof your love.

In a manner of speaking
I just want to say
That I could never forget the way
You told me everything
By saying nothing.

In a manner of speaking, Tuxedomoon

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Introduction

Une naissance commune, des vies distinctes

En se penchant sur l'histoire de la logique moderne, on peut constater que les logiques modales et les logiques multivaluées ont été introduites à la même époque. Mieux, il apparaît que certains logiciens, ŁUKASIEWICZ en particulier, définirent des systèmes multivalués avec pour premier but de pouvoir rendre compte de certaines modalités (voir [**26**]). L'ajout d'une troisième valeur de vérité permettait par exemple d'exprimer qu'une formule est possible sans être vraie.

Néanmoins, ces deux types de formalismes empruntèrent rapidement des chemins indépendants parce qu'ils se révélèrent être deux généralisations aux caractéristiques très distinctes du calcul propositionnel. D'une part, avec les logiques multivaluées comme définies par ŁUKASIEWICZ (voir [**40, 41, 42**]), le logicien peut choisir les valeurs de vérité de ses variables propositionnelles dans un ensemble à plus de deux éléments. L'accent était donc ici mis du côté de la sémantique.

D'autre part, en enrichissant le langage propositionnel de nouveaux connecteurs, appelés modalités, le logicien a pour but de pouvoir nuancer chacune de ses formules : une formule peut être possible, connue, prouvable etc. L'accent était mis du côté syntaxique.

Dans les deux cas, l'approche algébrique de ces systèmes formels donna d'intéressants résultats. Ainsi, les logiques multivaluées de ŁUKASIEWICZ furent approchées au travers de la variété des MV-algèbres introduite par CHANG dans [**5**] et [**6**]. Cette approche algébrique permit par exemple à CHANG de donner une preuve algébrique du théorème de complétude de la logique infivaluée de ŁUKASIEWICZ (voir [**6**]). Depuis lors, cette variété n'a eu cesse d'attirer l'attention des algébristes pour ses multiples propriétés généralisant celles des algèbres de Boole (voir [**10**] et [**22**]).

L'approche algébrique des logiques modales fut quand à elle introduite par JÓNSSON et TARSKI dans [**33**] et [**34**]. La variété des algèbres de Boole à opérateurs qu'ils définirent fournit une sémantique vis à vis de laquelle toute logique modale normale est complète. Mais, comme nous allons le préciser, cette sémantique algébrique ne reçut qu'une vingtaine d'années plus tard toute l'attention qu'elle mérite.

Entre temps, les années soixante virent naître un type de sémantique qui fut responsable du succès des logiques modales chez les mathématiciens, les informaticiens, les philosophes et les linguistes. Il s'agit de la classe des sémantiques relationnelles. L'idée de base de cette approche est très attirante intuitivement. Un modèle de KRIPKE est un ensemble non vide W (qu'on appelle *univers* et dont les éléments sont appelés *mondes*) muni d'une relation binaire R et d'une valuation Val, c'est à dire d'une fonction qui associe à toute variable propositionnelle en tout monde une valeur de vérité dans $\{0, 1\}$. Cette valuation est étendue inductivement à l'ensemble des formules en utilisant les règles évidentes pour les connecteurs booléens. En ce qui concerne le connecteur $\Diamond$ de possibilité, la règle stipule que la formule $\Diamond\phi$ est vraie en un monde u s'il existe un monde v accessible à partir de u (c'est-à-dire tel que $(u, v) \in R$) en lequel la formule ϕ est vraie.

Les logiciens constatèrent rapidement qu'en ajoutant des conditions sur la relation d'accessibilité R, les modèles de KRIPKE fournissent une sémantique complète pour divers types de logiques modales normales (voir [**38**] pour les premiers résultats de complétude de KRIPKE ou [**1**] ou [**4**] pour une synthèse des résultats actuels). C'est ainsi que débuta l'étude systématique des liens entre les structures relationnelles et les logiques modales.

Une approche multiple des structures relationnelles

L'universalité des structures de KRIPKE permet d'aborder cette étude (de manière non exclusive) à l'aide de différents outils mathématiques. Parmi ceux-ci, notons la théorie des modèles, l'algèbre universelle et la théorie des coalgèbres dont l'application à l'étude des logiques modales est plus récente.

Une structure de KRIPKE peut être considérée comme un modèle pour un langage du premier ordre ne contenant qu'un unique symbole relationnel binaire. Certaines propriétés de ces structures peuvent donc être à la fois définies par des formules modales ou des formules du premier ordre. L'étude générale de ces problèmes de définissabilité et des correspondances entre langage du premier ordre et langage de la logique modale peut naturellement être abordé grâce à la théorie des modèles. Parmi les résultats célèbres obtenus par ce biais, citons les résultats de SAHLQVIST (voir [**50**]) qui caractérisent une famille de propriétés du premier ordre qui sont également définissables par des formules modales et qui donnent une traduction automatique de ces propriétés entre les deux langages. Notons également bien sûr les résultats de VAN BENTHEM (voir [**55**]) qui caractérisent la logique modale comme le fragment de la logique du premier ordre qui est invariant par bisimulation.

L'approche algébrique des structures de KRIPKE apparaît de manière déguisée dans les articles fondateurs [**33**] et [**34**] de la théorie des algèbres de Boole à opérateurs. Ce n'est que bien des années plus tard que les mathématiciens prirent pleinement conscience de la richesse des liens qui existent entre la sémantique relationnelle et la sémantique algébrique. Ces liens se matérialisent au travers de deux types de construction : la construction de l'algèbre complexe associée à une structure d'une part et la construction de la structure canonique associée à une algèbre de Boole à opérateurs d'autre part.

Très brièvement, l'algèbre complexe d'une structure $\mathfrak{F}$ basée sur l'univers W est l'algèbre de Boole des fonctions de W dans l'algèbre de Boole à deux éléments $\mathbf{2}$ sur laquelle est greffée une nouvelle opération dont le but est de traduire l'information contenue dans la relation d'accessibilité de $\mathfrak{F}$. L'algèbre complexe encapsule la théorie modale de $\mathfrak{F}$ puisqu'une formule $\phi \leftrightarrow \psi$ est valide dans $\mathfrak{F}$ si et seulement si l'équation correspondante $\phi = \psi$ est satisfaite dans l'algèbre complexe de $\mathfrak{F}$.

La structure canonique d'une algèbre de Boole à opérateur A a pour univers l'ensemble des homomorphismes de A dans $\mathbf{2}$ et pour relation d'accessibilité la plus grande relation compatible avec l'opérateur modal de A. Si une formule $\phi \leftrightarrow \psi$ est valide dans cette structure canonique alors l'équation $\phi = \psi$ correspondante est satisfaite dans l'algèbre à laquelle la structure est associée.

Les combinaisons de ces constructions permettent une traduction algébrique des questions concernant le couple langage modal - structure relationnelle. Elles permettent par exemple de résoudre des problèmes de complétude et d'incomplétude, de définissabilité etc.

Les résultats obtenus grâce à cette approche algébrique sont nombreux. Citons, parce que nous en considérons des généralisations dans cette thèse, les travaux de GOLDBLATT et THOMASON à propos de la définissabilité des classes élémentaires de structures (voir [**24**]), ceux de JÓNSSON (voir [**32**]) qui constituent une version algébrique de ceux de SAHLQVIST ainsi que les résultats qui étendent la dualité de STONE aux algèbres de Boole à opérateurs (voir [**29, 52**]).

Fusion des genres

Nous consacrons cette dissertation à une étude de certaines généralisations multivaluées des logiques modales. Certains auteurs ont déjà initié de telles généralisations (voir [**15**], [**13**], [**14**], [**48**]). Puisque c'est l'existence des sémantiques relationnelles qui donna à la logique modale ses lettres de noblesse, il a semblé cohérent à ces auteurs de considérer ce type de sémantique comme point de départ d'une approche multivaluée des logiques modales. Il s'agit donc de sauver la sémantique de KRIPKE afin de maximiser les chances de survie des logiques développées.

La variété des systèmes déjà introduits le prouve, cette contrainte laisse encore énormément de libertés dans le choix de la sémantique à adopter : il y a de nombreuses possibilités de généraliser la définition d'un modèle de KRIPKE à un cadre mutlivalué. Néanmoins, ces généralisations peuvent se répartir en deux classes, non disjointes. Il s'agit de la classe des modèles de KRIPKE dans lesquels les variables propositionnelles sont évaluées dans un ensemble à plus de deux valeurs de vérité et de la classe des modèles de KRIPKE dont la relation d'accessibilité est multivaluée.

Face à ces deux types de généralisations, le choix du logicien est dicté par différents critères : il peut s'agir des applications qu'il envisage pour ces systèmes formels (comme dans [**14**]), du thème des résultats qu'il désire obtenir en priorité (translation entre les formules modales et les formules du premier ordre par exemple), de leur portée ou de leur profondeur, des outils qu'il envisage d'appliquer (algèbres, coalgèbres, théorie des modèles), d'une intuition ou d'appétences.

Dans notre choix, nous avons été guidé par la volonté de considérer des modèles de KRIPKE multivalués pour lesquels les outils algébriques existant pouvaient être appliqués ou généralisés. C'est ainsi que nous avons décidé de reposer notre approche sur les logiques multivaluées de ŁUKASIEWICZ. Les modèles de KRIPKE que nous considérons sont ainsi des modèles dans lesquels les variables propositionnelles sont évaluées dans une sous MV-algèbre complète de la MV-algèbre $[0,1]$ (les relations ne sont pas multivaluées). Étant donné que la variété des MV-algèbres partage beaucoup de propriétés avec celle des algèbres de Boole, nous avions espoir de trouver dans cette variété les caractéristiques requises pour une approche algébrique menant au moins à un théorème de complétude.

Malheureusement, même si nous obtenons des résultats intéressants dans le cas le plus général des modèles $[0,1]$-valués, les résultats les plus forts que nous prouvons dans cette dissertation concernent les modèles $Ł_n$-valués (où $Ł_n$ désigne la sous-algèbre $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ de $[0,1]$ pour tout entier strictement positif n). Ainsi, par exemple, ce n'est que dans les cas

Ł_n-valués que nous parvenons à décrire la plus petite logique modale normale, c'est à dire l'ensemble des formules qui sont vraies dans tous les modèles Ł_n-valués. La difficulté d'obtenir des résultats pour les modèles $[0,1]$-valués est imputable à certains défauts des théorèmes de représentation des MV-algèbres. En effet si, comme c'est le cas pour les algèbres de Boole, une MV-algèbre d'une variété finiment engendrée peut être représentée comme un produit booléen de ses quotients simples, ce résultat n'est pas vrai dans la variété des MV-algèbres (dans laquelle les filtres premiers et maximaux ne coïncident pas).

Le contenu

Prolégomènes. Le premier chapitre de cette dissertation rappelle quelques notions à propos de la variété des MV-algèbres. Le choix de ces notions est forcément orienté, et le lecteur qui désire plus d'informations à propos de cette variété est invité à consulter la monographie [**10**] ou l'article [**22**]. Outre des généralités, nous rappelons aussi dans la troisième section de ce chapitre quelques résultats à propos de la construction de certains termes du langage des MV-algèbres. Ces résultats sont d'une importance capitale pour le reste de la dissertation.

Modèles et structures. Le deuxième chapitre débute par l'introduction des modèles de Kripke A-valués où A est une MV-algèbre complète. Nous prouvons que si A est une MV-algèbre complète et complètement distributive, alors dans un certain sens, tout problème concernant les modèles A-valués équivaut à un problème à propos de modèles $[0,1]$-valués. Nous introduisons également les modèles pour une généralisation multivaluée de la logique dynamique propositionnelle qui est une logique de programmes qui repose sur une interprétation des programmes dans le langage modal. Nous illustrons ces définitions en montrant qu'il est possible de rajouter une « couche dynamique » à l'interprétation du jeux de Rényi - Ulam développée dans [**45**] en termes de MV-algèbres et de logiques finivaluées de Łukasiewicz.

Nous introduisons alors un deuxième niveau dans la gamme des sémantiques relationnelles : le niveau des structures. Une structure de Kripke est un ensemble muni d'une relation binaire (un modèle sans valuation). Une formule est valide dans une structure de Kripke si elle est vraie dans tous les modèles obtenus en rajoutant une valuation à la structure. Il s'agit donc du niveau adéquat pour l'étude des structures relationnelles à l'aide du langage modal.

Nous définissons alors un niveau supplémentaire, celui des p-structures et des Ł_n-structures. Il s'agit de structures de Kripke dans lesquelles on précise, pour chaque monde, l'ensemble des valeurs de vérités que les formules sont autorisées à prendre en ce monde (cet ensemble est une sous-algèbre complète de $[0,1]$). Les valuations qui sont autorisées sur ces structures doivent respecter ces règles. La classe des Ł_n-structures est une sous-classe élémentaire de la classe des p-structures. Quant à celle-ci, elle est obtenue comme une classe élémentaire dans l'extension du langage des structures de Kripke qui contient un prédicat unaire r_B pour toute sous-algèbre complète B de $[0,1]$. La relation r_B contient les mondes qui évaluent leurs formules dans B.

À ce stade des constructions, le lecteur ne réalise peut-être pas l'importance de ces classes de structures dans la gamme des sémantiques relationnelles pour les logiques modales multivaluées. En fait, il s'agit du niveau le plus naturel pour l'étude algébrique des liens entre logiques modales multivaluées et sémantique relationnelles. En effet, la théorie des extensions canoniques peut être appliquée pour étudier des problèmes de complétude de logiques $Ł_n$-valuées, mais les résultats obtenus impliquent des classes de $Ł_n$-structures et non simplement des classes de structures de Kripke.

Le deuxième chapitre se poursuit par la présentation d'une panoplie de constructions de structures auxquelles sont associées des résultats de préservation de validité de formules du langage modal. Ces constructions permettent d'emblée de donner des exemples de propriétés de structures ou de $Ł_n$-structures qui ne peuvent pas être définies par des formules modales parce qu'elles ne sont pas conservées par une de ces constructions. De plus, elles apparaîtront dans la version $Ł_n$-valuée du théorème de Golblatt - Thomason qui caractérise ces classes modalement définissables.

L'outil algébrique entre alors en jeu par l'intermédiaire de la variété des MV-algèbres à opérateurs que nous définissons. Nous introduisons également diverses notions d'algèbres complexes associées à des structures (ou p-structures) afin de capturer dans le langage algébrique les théories modales de ces structures.

Les constructions inverses, celles qui permettent d'associer des structures aux algèbres sont alors envisagées. Ainsi, si A est une MV-algèbre à opérateurs, ses structures canoniques ont pour univers l'ensemble des filtres maximaux de A. À cet égard, le Lemme 2.40 est fondamental pour la suite de la dissertation. Il prouve que le modèle canonique associé à un modèle algébrique s'étend naturellement aux formules.

L'occasion se présente alors de prouver que les différents types de constructions de structures que nous avons précédemment introduits possèdent un correspondant algébrique.

Nous généralisons ensuite dans la cinquième section du deuxième chapitre la célèbre dualité entre la catégorie des algèbres de Boole complètes et complètement distributives avec opérateurs complets d'une part (dont les flèches sont les homomorphismes complets) et la catégorie des structures de Kripke (dont les flèches sont les morphismes bornés) à des dualités pour des sous-catégories complètes de la catégorie des MV-algèbres complètes et complètement distributives d'une part et des catégories de structures.

Nous considérons ensuite un dernier type de construction de structures. Il s'agit de l'extension canonique, obtenue en composant la construction de l'algèbre complexe d'une structure avec celle de la structure canonique d'une algèbre.

Enfin, nous obtenons, en suivant les traces des résultats des auteurs, deux généralisations du théorème de Goldblatt - Thomsason. En effet, les Théorèmes 2.75 et 2.78 caractérisent respectivement les classes de $Ł_n$-structures et de structures fermées par ultraproduit qui sont $Ł_n$-modalement définissables.

Systèmes modaux multivalués et complétude. Nous consacrons le troisième chapitre au problème général de complétude des logiques modales vis à vis des classes de structures relationnelles. Les résultats s'obtiennent par l'intermédiaire de la sémantique algébrique. En effet, toute logique modale normale multivaluée est complète vis à vis de la variété des algèbres qu'elle définit.

Cette complétude algébrique peut dans certains cas être traduite en un résultat de complétude vis à vis des sémantiques de Kripke via les constructions des structures canoniques et des algèbres complexes. À cet effet, le Lemme 2.40 déjà mentionné joue un rôle d'une importance capitale.

C'est ainsi que nous obtenons une description syntaxique de la plus petite logique modale $Ł_n$-valuée $\mathbf{K}_n$, c'est-à-dire de l'ensemble des formules qui sont vraies dans tous les modèles $Ł_n$-valués. Nous obtenons également un tel résultat pour les modèles $Ł_n$-valués de la logique propositionnelle dynamique $Ł_n$-valuée.

Malheureusement, nous n'obtenons qu'un résultat partiel pour la description de la plus petite logique modale $[0,1]$-valuée $\mathbf{K}$, c'est à dire l'ensemble des formules qui sont vraies dans tout modèle $[0,1]$-valué. En effet, dans le résultat que nous obtenons intervient une règle d'inférence non finie : cette règle nous amène à spécifier que ϕ est un théorème si nous pouvons prouver que $\phi \oplus \phi^n$ est un théorème pour tout naturel non nul n. Le problème de l'axiomatisation de $\mathbf{K}$ sans une telle règle infinie reste ouvert.

Nous introduisons alors deux types de canonicités : la canonicité et la canonicité forte. À l'aide de leur traduction algébrique, ces deux types de canonicité permettent d'obtenir des logiques modales $Ł_n$-valuées qui sont complètes vis à vis de classes des structures (pour la canonicité forte) et de $Ł_n$-structures (pour la canonicité). Nous obtenons par exemple que les classes de structures et $Ł_n$-structures élémentaires ont des théories modales complètes (voir Théorème 3.47) généralisant un résultat bien connu.

Canonicité dans $\mathcal{MMV}_n^{\mathcal{L}}$: une chemin syntaxique. Obtenir des variétés canoniques et fortement canoniques permet donc de définir des logiques complètes. Une des méthodes les plus fécondes pour la construction de telles variétés consiste à étudier la conservation d'équations au travers de ces constructions. Un ensemble d'équations conservées par extension canonique (forte) définit une variété (fortement) canonique. Cette approche syntaxique de la canonicité fut initiée dans [**33**] et [**34**] pour les algèbres de Boole à opérateurs pour être pleinement exploitée dans [**32**] afin de donner une preuve algébrique des résultats de complétude de Sahlqvist. Depuis généralisée à la classe des expansions de treillis distributifs bornés [**20**], puis des expansions des treillis bornés [**17**], cette approche de la canonicité a déjà produit d'intéressants résultats (voir [**21**] par exemple).

Comme il est possible de considérer la variété des MV-algèbres comme une variété d'expansions de treillis distributifs bornés, notre approche de la canonicité repose sur les résultats de [**20**]. La variété des MV-algèbres n'étant pas canonique (voir [**16**]), nous restreignons notre étude aux variétés de MV-algèbres à opérateurs dont la MV-algèbre sous-jacente appartient à une variété finiment engendrée.

Après avoir rappelé les résultats de [**20**], nous prouvons que la définition d'extension canonique d'une MV_n-algèbre à opérateurs que nous avons adoptée dans le chapitre 2 coïncide avec la définition classique (c'est-à-dire selon l'approche de [**20**]). Il s'agit d'un résultat essentiel puisqu'il permet de connecter l'approche classique de la canonicité avec les sémantiques de Kripke pour les logiques modales $Ł_n$-valuées.

Ensuite, nous pouvons adopter la technique classique pour étudier la stabilité des équations au travers de ces extensions. Cette technique repose sur l'existence de liens entre les propriétés de stabilité des termes et les propriétés de continuité des connecteurs qui les composent.

En mimant la démonstration du résultat pour les logiques modales bivaluées, nous obtenons le correspondant du théorème de Sahlqvist pour les logiques modales $Ł_n$-valuées et les extensions canoniques. Ce résultat corrobore notre point de vue sur l'approche algébrique des sémantiques relationnelles pour les logiques modales $Ł_n$-valuées : cette approche est adaptée à l'étude des $Ł_n$-structures plutôt que des structures.

Pour étudier la canonicité forte des logiques modales $Ł_n$-valuées, aucun outil existant ne pouvait être appliqué. Bien sûr, la définition de l'extension canonique forte d'une MV_n-algèbre à opérateurs A s'impose d'elle même : elle s'obtient en considérant la MV_n-algèbre complexe associée à la structure canonique de A. Cette construction peut être caractérisée à isomorphisme près comme la plus grande extension de A dont l'algèbre d'idempotents est isomorphe à l'extension canonique de l'algèbre d'idempotents de A.

Nous exploitons cette description pour étendre une application définie entre deux algèbres A et B en une application entre leurs extensions canoniques fortes. En effet, certaines applications entre MV_n-algèbres, que nous appelons idemorphismes, sont entièrement caractérisées par les valeurs qu'elle prennent sur les algèbres d'idempotents. Ainsi, si $f : A \to B$ est une telle application, nous pouvons définir f^τ sur l'extension canonique forte A^τ de A comme l'unique idemorphisme dont la restriction à l'algèbre de Boole d'idempotents de A^τ est l'extension canonique de la restriction de f à l'algèbre d'idempotents de A.

En transportant les propriétés de stabilité des termes au travers de l'extension canonique dans l'univers des extensions canoniques fortes, nous sommes amenés à prouver le Théorème 4.61 qui est un équivalent du théorème de SAHLQVIST pour les extensions canoniques fortes.

Une dualité topologique pour la catégorie $\mathcal{MMV}_n^{\mathcal{L}}$. L'extension canonique n'a pas toutes les qualités. En effet, cette construction fournit un théorème de représentation imparfait : cette extension ne satisfait peut-être pas toutes les équations satisfaites par l'algèbre de départ.

De l'information s'est donc perdue dans la construction. Comme pour les algèbres de Boole à opérateurs, l'information se perd au niveau de la construction de l'algèbre complexe : considérer l'ensemble des valuations possibles sur la structure canonique d'une algèbre A est trop général pour retrouver exactement le contenu algébrique de A.

Il est possible de remédier à ce défaut en ajoutant une nouvelle couche, de nature topologique, aux structures canoniques. Dans l'aventure, nous quittons donc le monde des structures définissables au premier ordre.

Pour ajouter cette couche, nous fusionnons deux types de constructions : la construction d'une dualité forte (en sens des dualités naturelles, voir [**11**]) pour la catégorie des MV_n-algèbres et la construction de la $Ł_n$-structure canonique associée à une MV_n-algèbre à opérateurs.

Nous obtenons alors une dualité ente la catégorie $\mathcal{MMV}_n^{\mathcal{L}}$ des MV_n-algèbres à opérateurs et une classe de structures topologiques. Cette dualité étend à $\mathcal{MMV}_n^{\mathcal{L}}$ la dualité de STONE pour les algèbres de Boole à opérateurs.

Ceci nous fournit bien sûr un résultat de complétude pour toute logique modale finivaluée **L**. Enfin, nous envisageons le problème de la construction des coproduits dans la catégorie duale.

Modalités

Pour terminer, précisons que, si dans cette introduction nous n'avons considéré que des logiques modales dans un langage ne contenant qu'un seul opérateur unaire, nous proposons dans cette dissertation des résultats plus généraux puisque nous autorisons autant de modalités d'arité quelconque que désiré.

Introduction

Born together, live apart

When one looks backwards in the history of modern logic, one can notice that modal logics and many-valued logics are born approximatively at the same time (see [**26**]). It even appears that some logicians, such as ŁUKASIEWICZ, defined many-valued systems in order to deal with modalities. By considering a third truth value, they meant to express that a formula can, for example, be possible without being true.

Nevertheless, these two types of formalisms followed their own ways. They are indeed two generalizations of propositional calculus with very different properties. On the one hand, with many-valued logics as defined, *e.g.*, by ŁUKASIEWICZ (see [**40, 41, 42**]), the logician can choose his truth values in a set containing more than two elements. The focus is set on the semantic side.

On the other hand, by enriching the propositional language with new connectives, called modalities, the logician aims to modify the meaning of his formulas: a formula can be possible, known, *etc.* The focus is set on the syntactic side.

In both cases, the algebraic approach of these formal systems brought lots of interesting results. Hence, ŁUKASIEWICZ many-valued logics were studied through the variety of MV-algebras that was defined by CHANG in [**5**] and [**6**]. This approach lead for example to an algebraic proof of the completeness result for ŁUKASIEWICZ' infinite-valued logic (see [**6**]). Ever since CHANG introduced this variety, the algebraist s studied it extensively as an extension of the variety of boolean algebras that has a lot of interesting properties (see [**10**] and [**22**]).

The algebraic approach of modal logics was initiated by JÓNSSON and TARSKI in their seminal papers [**33**] and [**34**]. They defined the variety of boolean algebras with operators that provides an algebraic semantic with respect to which every normal modal logic is complete. But, as we are going to realize, this semantic did not receive the attention it deserves for about twenty years.

Meanwhile, a new type of semantics for modal logics, called relational semantics, appeared during the sixties. This class of semantics is responsible for the success of modal logic in the areas of mathematic, computer science, linguistic and philosophy. The idea that underlies this approach is very appealing and intuitive. A KRIPKE model is given by a nonempty set W (called the *universe*) whose elements are called *worlds* together with a binary accessibility relation R on W and a valuation map Val, *i.e.*, a map that assigns a truth valued in $\{0, 1\}$ to any propositional variable p in a world w. This map is extended to formulas by following the obvious rules for boolean connectives. For the modality $\Diamond$, the rule specifies that the formula

$\Diamond\phi$ (read "ϕ is possible") is true in a world u if there is a world accessible from u in which the formula ϕ is true, i.e., the truth value $\mathrm{Val}(u, \Diamond\phi)$ is defined as $\bigvee\{\mathrm{Val}(v, \phi) \mid (u, v) \in R\}$.

Logicians realized that they can obtain completeness results for various normal modal logics and KRIPKE models by adding conditions on the accessibility relation R (see [**38**] for the first completeness results and [**1**] or [**4**] for surveys of up to date results). This was the birth of the in-depth study of the links that connect relational structures and modal logics.

Many-sided approach of relational structures

The universal aspect of KRIPKE structures allows to approach them with the help of numerous mathematical tools. Among them, let us cite model theory, universal algebra, category and coalgebra theory.

A KRIPKE frame can be seen as a first order model for a language containing a single binary relational symbol. Among the properties of these structures, some of them can be defined both by first order and modal formulas. The problem of definability and correspondence between first order formulas and modal ones can obviously be tackled by a model theoric approach. Among the results obtained in this way, let us cite SAHLQVIST's results that characterize a family of first order properties that can also be defined by modal formulas and that also provide an automatic translation of these properties between the two languages (see [**50**]). Let us also cite VAN BENTHEM's results that describe modal logic as the bisimulation invariant fragment of first order logic (see [**55, 56**]).

The first steps in the algebraic approach of KRIPKE frames already appeared in a disguised form in the seminal works [**33**] and [**34**] about boolean algebras with operators. Only many years later, mathematicians realized how deep the connections are between relational and algebraic semantics. These connections materialize through two types of constructions: the construction of the complex algebra associated to a frame on the one hand and the construction of the canonical frame associated to a boolean algebra with operators on the other hand.

Roughly speaking, the complex algebra of a frame $\mathfrak{F}$ with universe W is the boolean algebra of functions from W to the two element boolean algebra $\mathbf{2}$ on which a new operation is added in order to grab the information of the accessibility relation of $\mathfrak{F}$. The complex algebra contains the modal theory of $\mathfrak{F}$ since a formula $\phi \leftrightarrow \psi$ is valid in $\mathfrak{F}$ if and only if the corresponding equation $\phi = \psi$ is satisfied in the complex algebra of $\mathfrak{F}$.

The universe of the canonical frame associated to a boolean algebra with operator A is the set of boolean homomorphisms from A to $\mathbf{2}$ and its accessibility relation is the biggest relation compatible with the modal operator of A. The canonical structure only validates formulas $\phi \leftrightarrow \psi$ whose corresponding equation $\phi = \psi$ is satisfied in A.

The compositions of these constructions allow an algebraic translation of questions about the connections between modal language and relational structures. For example, completeness, incompleteness and definability can be tackled with these tools.

Among the results obtained thanks to this algebraic approach, let us cite GOLDBLATT and THOMASON's results (that we generalize in this dissertation) about modal definability of elementary classes of structures (see [**24**]). Let us also cite JÓNSSON's results (see [**32**]) which are an algebraic version of SAHLQVIST's completeness results and finally the results that extend STONE duality to boolean algebras with operators (see [**29, 52**]).

Fusion of style

This dissertation is focused on some many-valued generalizations of modal logics. Many authors have already initiated such studies (see **[15]**, **[13]**, **[14]**, **[48]**). As each of these authors realized, since the success of modal logics is a consequence of their KRIPKE semantics, it is wise to consider this semantics as a starting point for many-valued generalizations of modal logics. In other words, let us keep KRIPKE semantic to optimize the survival rate of these new many-valued modal logics.

The diversity of the many-valued modal systems that have already been introduced proves that the principle of "keeping KRIPKE semantic" still allows a lot of freedom in the definitions. Indeed, there are many ways in which one can generalize KRIPKE models to a many-valued realm. Nevertheless, these generalizations can be classified in two (non exclusive) classes: the class of the KRIPKE models in which propositional variables are evaluated in a set with more than two elements and the class of the KRIPKE models in which the accessibility relation is many-valued.

Facing these possibilities, the logician may combine several criteria to determine the approach he want to follow. His choice can be guided by the applications he wishes to develop for his systems (as in **[14]**), by the theme of the results that are to be obtained in priority (translation between modal formulas and first order formulas for example), by the tools he wishes to apply (algebras, coalgebras, model theory, ...), by his intuition and his abilities.

In our case, we were guided by the will to consider many-valued KRIPKE models for which the existing algebraic tools could be applied or generalized. Hence, we have decided to base our approach on ŁUKASIEWICZ logic. The KRIPKE models that we consider are models in which variables have their truth value in a complete subalgebra of the MV-algebra $[0,1]$ (relations are not many-valued). Since the variety of MV-algebras shares a lot of properties with the variety of boolean algebras, we hoped to find in this variety the properties required for an algebraic approach that would lead, at least, to a completeness result.

Unfortunately, even if we obtain interesting results in the general case of $[0,1]$-valued models, the strongest results we could prove are about $Ł_n$-valued models (where $Ł_n$ is the subalgebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$ for any strictly positive integer n). Hence, for example, we are able to describe the smallest A-valued modal logic, *i.e.*, the set of formulas that are true in any A-valued KRIPKE-model, only of A is equal to $Ł_n$ for a strictly positive integer n. The difficulty in getting results for $[0,1]$-valued models is a consequence of some weaknesses in the representation theorems for MV-algebras. Indeed, as in the case of boolean algebras, any MV-algebra that belongs to a finitely generated variety can be represented as a subdirect product of its simple quotients, but this results is not true in the whole variety of MV-algebras (in which prime filters and maximal filters do not coincide).

The content of the dissertation

Prolegomena. In the first chapter of this dissertation, we recall some definitions and results about MV-algebras. The style may seem rough and the lost reader may consult the monograph **[10]** or the paper **[22]** to obtain complementary information about this variety. Besides the necessary general results, we recall in section 3 of chapter 1 some (folklore) results about the constructions of some terms in the language of MV-algebras. These results are widely applied in the entire dissertation.

Models and structures. We start the second chapter by introducing the A-valued KRIPKE models where A is a complete MV-algebra. We prove that if A is a complete and completely distributive MV-algebra then, in a way, any problem about A-valued models is equivalent to a problem about $[0,1]$-valued models. We also introduce the models for a many-valued generalization of propositional dynamic logic which is a logic of programs that is build upon an interpretation of programs in the modal language. We illustrate these definitions by showing that it is possible to add a "dynamic layer" to the interpretation in terms of MV-algebras and ŁUKASIEWICZ finitely-valued logics of the RÉNYI - ULAM's game that is developed in [**45**].

Then we define a second level in the class of relational semantics: the level of frames. A KRIPKE frame is given by a set together with a binary relation on that set (*i.e.*, a frame is a model without valuation). A formula is valid in a frame if it is true in any model obtained by adding a valuation to the frame. Thus, the level of frames is the adequate level for the approach of relational structures with the modal language.

Another level is introduced: the level of p-frames and $Ł_n$-frames. These are frames in which we specify in every world the set of allowed truth values for the formulas in that world (this set is a complete subalgebra of $[0,1]$). Any valuation added to the frame has to respect these rules. The class of the $Ł_n$-frames is an elementary subclass of the class of the p-frames. The class of the p-frames is defined as an elementary class in the extension of the language of frames that contains a unary predicate r_B for any complete subalgebra B of $[0,1]$. By definition, the relation r_B contains the worlds that evaluate their formulas in B.

At this step, the reader may not realize how important is this level in the family of relational semantics for many-valued modal logics. The level of $Ł_n$-frames is actually the more natural one for an algebraic approach of the connections between $Ł_n$-valued modal logics and relational semantic. Indeed, the theory of canonical extension can be applied to obtain completeness results for $Ł_n$-valued modal logics, but these results involve classes of $Ł_n$-frames and not simply classes of frames.

The second chapter continues with the introduction of several types of constructions of structures. Each of this type of construction preserves validity of modal formulas in a way or another. Thus, these are tools that can be used to provide examples of frames or p-frames properties that cannot be defined by modal formulas since they are not preserved by one of these constructions. Moreover, they appear in the $Ł_n$-valued counterpart of the GOLBLATT - THOMASON theorem that characterizes modally definable classes of frames and $Ł_n$-frames that are closed under ultraproducts.

The algebraic tool comes then into the picture through a suitable variety of MV-algebras with operators. We also introduce several notions of complex algebras that we associate to frames (or p-frames) in order to capture the modal theory of these structures in the algebraic language.

The reverse constructions that associate structures to algebras are then considered. If A is an MV-algebra with operators, the universe of its canonical structures is the set of the maximal filters of A. In this respect, Lemma 2.40 is an essential result for the dissertation. This result proves that the canonical model associated to an algebraic model extends naturally to formulas.

Then, we take the opportunity to show that the different types of constructions of structures that we have previously introduced have an algebraic counterpart.

In section 4 of chapter 2, we generalize the famous duality between the category of complete and completely distributive boolean algebras with complete operators (whose arrows are the complete homomorphisms) and the category of KRIPKE frames (whose arrows are the bounded morphisms). We obtain dualities for some complete subcategories of the category of complete and completely distributive MV-algebras and some categories of structures (frames and p-frames).

We continue by introducing a last type of construction of structures, namely, the canonical extension. This extension is obtained as the canonical structures of the complex algebra of a structure.

Eventually, we obtain two generalizations of the GOLDBLATT - THOMSASON theorem by mimicking the original proof. Our results characterize the classes of frames and $Ł_n$-frames closed under ultraproducts that are $Ł_n$-modally definable.

Many-valued modal systems and completeness. The third chapter is dedicated to the problem of completeness of many-valued modal logics with respect to relational semantics. The results are obtained through the algebraic semantic. Indeed, any many-valued normal modal logic is complete with respect to the variety of algebras that it defines.

These algebraic completeness results can, in some cases, be translated into relational completeness results thanks to the construction of the canonical structures and complex algebras. Lemma 2.40 plays a key role to that aim.

In that way, we obtain a syntactic description of the smallest normal modal $Ł_n$-valued logic $\mathbf{K}_n$, which is the set of formulas that are true in every $Ł_n$-valued KRIPKE model. We also prove such a result for $Ł_n$-valued models of propositional $Ł_n$-valued dynamic logic.

Unfortunately, we only obtain partial results for the description of the smallest normal modal $[0,1]$-valued logic $\mathbf{K}$ which is the set of formulas that are true in any $[0,1]$-valued KRIPKE model. Indeed, in our results, we use an infinitary deduction rule. This rule states that we have to accept that ϕ is a theorem whenever we find that $\phi \oplus \phi^n$ is a theorem for every strictly positive integer n. The problem of the axiomatization of $\mathbf{K}$ without such an infinitary rule is still open (see [**2**]).

We also introduce two types of canonicity: the canonicity and the strong canonicity. Thanks to their algebraic translation, these two types of canonicity allow to obtain $Ł_n$-valued modal logics that are complete with respect to classes of frames (strongly canonical logics) or with respect to classes of $Ł_n$-frames (canonical logics). Hence, we obtain for example that elementary classes of frames and $Ł_n$-frames have a modal theory that is complete (see Theorem 3.47).

Canonicity in $\mathcal{MMV}_n^{\mathcal{L}}$: a syntactic approach. Obtaining canonical and strongly canonical varieties helps to define complete logics. One of the most fruitful methods to construct such varieties is a syntactical one: any variety that is defined by equations that are preserved by (strong) canonical extensions is a (strongly) canonical variety. This approach was initiated in [**33**] and [**34**] for varieties of boolean algebras with operators and was fully developed in [**32**] in which an algebraic proof of SAHLQVIST's canonicity result is given. Since then, the theory of canonical extensions was generalized to bounded distributive lattice expansions in [**20**], then to bounded lattice expansions in [**17**]. It is nowadays recognized for its applications (*e.g*, [**21**]).

Since the variety of MV-algebras can be considered as a variety of bounded distributive lattice expansions, our approach of canonicity is based on the results of [**20**]. As the whole variety of MV-algebras is not canonical (see [**16**]), we focus our work on varieties of MV-algebras with operators whose MV-reduct belongs to a finitely generated variety.

We first recall the results of [**20**]. Then, we prove that the definition of the canonical extension of an MV$_n$-algebra with operators that we have adopted in the second chapter coincides with the classical definition (*i.e.*, the definition of [**20**]) for which an algebra is considered as a bounded distributive lattice expansion. This result is fundamental since it allows to connect the classical approach of canonicity with KRIPKE semantics for Ł$_n$-valued modal logics.

Afterwards, we apply and adapt the classical techniques to study stability of equations through canonical extensions. These techniques are based on the links between stability properties of term functions and continuity properties of the interpretation of their connectives.

Hence, we obtain a SAHLQVIST equivalent for Ł$_n$-valued modal logics by mimicking the algebraic proof of the corresponding result for classical modal logic. This result sustains our claim that the algebraic approach of relational semantics for Ł$_n$-valued modal logics fits more the class of Ł$_n$-frames than the class of frames.

In order to study strong canonicity for Ł$_n$-valued modal logics, no existing tool could be applied. The definition of the strong canonical extension of an MV$_n$-algebra with operators A is prescribed by the desired applications: this extension is defined as the Ł$_n$-complex algebra associated to the canonical frame of A. This construction can be characterized up to isomorphism as the greatest extension of A whose algebra of idempotent elements is isomorphic to the canonical extension of the algebra of idempotent elements of A.

We use that description to define the extension of some maps between two algebras A and B to maps between their strong canonical extensions. Indeed, there is a class of applications (that we call *idemorphisms*) between MV$_n$-algebras that are characterized by their restriction to the algebra of idempotent elements. Hence, if $f : A \to B$ is such an application, we can define f^τ on the strong canonical extension A^τ of A as the unique idemorphism whose restriction to the algebra of idempotent elements of A^τ is the canonical extension of the restriction of f to the algebra of idempotents of A.

By transporting the properties of stability of terms through canonical extensions into the realm of strong canonical extensions, we are able to prove Theorem 4.61 which is an equivalent of SAHLQVIST's canonicity result for strong canonical extensions.

A topological duality for the category $\mathcal{MMV}_n^{\mathcal{L}}$. The theory of canonical extensions provides an imperfect representation result: the canonical extension of an algebra A may not satisfy every equation that is satisfied in A.

Some information has been lost in the process. Similarly to boolean algebras with operators, the information is lost when the Ł$_n$-tight complex algebra of the canonical Ł$_n$-frame of A is constructed: the algebra of all the possible valuations on that structure is in general too wide to embody exactly the algebraic content of A.

It is possible to compensate for the missing information by adding a new topological layer to canonical structures. We thus leave the world of the structures that are definable by first order formulas.

We combine two types of constructions in order to obtain the desired layer. The first one is the construction of a strong duality (in the sense of natural duality, see [**11**]) for the category of MV_n-algebras and the construction of the canonical $Ł_n$-frame associated to an MV_n-algebra with operators.

In that way, we obtain a duality between the category $\mathcal{MMV}_n^{\mathcal{L}}$ of MV_n-algebras with operators and a category of topological structures. This duality is an extension of the Stone duality for boolean algebras with operators.

Of course, this duality provides a completeness result for any $Ł_n$-valued normal modal logic $\mathbf{L}$. We eventually consider the problem of the construction of the coproducts in the dual category.

Modalities

To conclude, let us precise that, if in this introduction we have only considered modal logics in a language that contains a single unary modality, the framework of this dissertation is more general since we allow in the language any set of modalities of any finite arity.

CHAPTER 1

Prolegomena

We provide in this chapter the building blocks of the results developed in this dissertation. We assume that the reader is familiar with the theory of universal algebra (see [**3**] and [**27**]) and general topology, that he knows the basic vocabulary of category theory (see [**43**]), and that he has already been introduced to the algebraic treatment of (modal or many-valued) logics (see [**1**], [**10**], [**28**]).

1. MV-algebras

MV-algebras were introduced by C.C. Chang (cf. [**5**] and [**6**]) as an algebraic counterpart of Łukasiewicz's many-valued logics (see [**40**] and [**41**]). Lindenbaum algebras of many-valued logics are indeed MV-algebras and this algebraic approach lead, for example, to an algebraic proof of the completeness of Łukasiewicz infinite-valued logic (see [**6**]).

From then on, the variety of MV-algebras, which is an extension of the variety of boolean algebras, and its connections with other areas of mathematics were studied by various mathematicians with various aims. The reader may consult [**10**] and [**22**] to obtain proofs and references for the results we state in this section.

Since 1958, the variety of MV-algebras has been given several equational bases (in the literature, MV-algebras are sometimes called Wajsberg algebras, bounded commutative BCK-algebras and you may consider them as a subvariety of the variety of BL-algebras). Following recent authors, we prefer the following definition.

Definition 1.1. An algebra $A = \langle A, \oplus, \odot, \neg, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ is an *MV-algebra* if $\langle A, \oplus, 0\rangle$ is an abelian monoid and if A satisfies the following equations :

(1) $\neg\neg x = x$,
(2) $x \oplus 1 = 1$
(3) $\neg 0 = 1$,
(4) $x \odot y = \neg(\neg x \oplus \neg y)$,
(5) $(x \odot \neg y) \oplus y = (y \odot \neg x) \oplus x$.

We denote by $\mathcal{MV}$ the variety of MV-algebras. If A is an MV-algebra, we denote by $\rightarrow$ the operation defined on A by

$$x \rightarrow y = y \oplus \neg x.$$

For convenience's sake, as it appears in this definition, we do not distinguish in the notations an algebra from its universe and we do not, in general, distinguish an operation symbol with its interpretation on an algebra.

Lemma 1.2. *If A is an MV-algebra, the relation $\leq$ defined on A by*

$$x \leq y \quad \textit{if} \quad x \rightarrow y = 1$$

is a bounded distributive lattice order on A (the lower bound is 0 et the upper bound is 1). The associated lattice operations $\vee$ and $\wedge$ are obtained in the following way :

$$\begin{aligned} x \vee y &= (y \odot \neg x) \oplus x \\ x \wedge y &= (y \oplus \neg x) \odot x. \end{aligned}$$

Moreover, the operations $\oplus$ and $\odot$ distributes over $\vee$ and $\wedge$.

Of course, an *MV-chain* is an MV-algebra whose lattice order is a total order. In any MV-algebra, you can find a privileged boolean algebra.

DEFINITION 1.3. If A is an MV-algebra, an *idempotent element of A* is an element x of A such that $x \oplus x = x$. The *algebra of idempotents of A* is the subalgebra

$$\mathfrak{B}(A) = \{x \in A \mid x \oplus x = x\}$$

of A.

One can prove that $\mathfrak{B}(A)$ is the largest subalgebra of an MV-algebra A which is a boolean algebra. Moreover, the variety of boolean algebras can be obtained by adding the equation $x \oplus x = x$ to the equational base of the variety of MV-algebras.

EXAMPLE 1.4. We present a few important examples of MV-algebras.

(1) The algebra $[0,1] = \langle [0,1], \oplus, \odot, \neg, 0, 1\rangle$ where $[0,1]$ is the real unit interval and where $x \oplus y = \min(x+y, 1)$ and $\neg x = 1 - x$ is an MV-algebra. We sometimes denote by $Ł_0$ this MV-algebra.
(2) For any positive integer n, the set $Ł_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ is a subalgebra of $[0,1]$.
(3) The algebra $\mathcal{C} = \langle C, \oplus, \odot, \neg, (0,0), (1,0)\rangle$ of CHANG is defined on

$$C = \{(0,a) \mid a \in \mathbb{Z}^+\} \cup \{(1,b) \mid b \in \mathbb{Z}^-\},$$

by

$$(i,x) \oplus (j,y) = \begin{cases} (0, x+y) & \text{if } i+j=0 \\ (1, \min(0, x+y)) & \text{if } i+j=1 \\ (1,0) & \text{if } i+j=2 \end{cases}$$

and

$$\neg(i,x) = \begin{cases} (0,-x) & \text{if } i=1 \\ (1,-x) & \text{if } i=0. \end{cases}$$

The algebra $\mathcal{C}$ is an MV-algebra.

We give a glimpse of the properties of the the variety $\mathcal{MV}$ and of its subvarieties.

THEOREM 1.5. *The variety $\mathcal{MV}$ is generated by the algebra $[0,1]$. The finitely generated subvarieties of $\mathcal{MV}$ are exactly the varieties generated by a finite number of finite subalgebras $Ł_n$ of $[0,1]$. Moreover, the algebra $\mathcal{C}$ of* CHANG *belongs to any non-finitely generated subvariety of $\mathcal{MV}$.*

Note that the completeness result for ŁUKASIEWICZ's infinite valued logic appears as a consequence of the first statement of the previous Theorem.

A complete classification of the subvarieties of $\mathcal{MV}$ was obtained by KOMORI in [**36**].

1.1. Congruences and implicative filters. As for boolean algebras, any congruence Θ of an MV-algebra is characterized by $1/\Theta$ (the class of 1 for Θ).

DEFINITION 1.6. Assume that A is an MV-algebra. A subset F of A is an *implicative filter* (or simply a *filter*) if F contains 1 and if F contains y whenever F contains x and $x \to y$.

As usual, a filter F is a *proper filter* if it does not contain 0 and a *non trivial filter* if $F \neq \{1\}$. A *maximal filter* is a filter which is maximal (for inclusion) among the proper filters. We denote by $\mathrm{Max}(A)$ the set of maximal filters of A.

The *distance function* on A^2 is defined by

$$d : A^2 \to A : (x, y) \mapsto (x \odot \neg y) \oplus (y \odot \neg x).$$

A filter F of A is a *prime filter* of A if for any x and y in A, either $x \to y$ or $y \to x$ belongs to A.

One may equivalently defines a filter of an MV-algebra A as an increasing non-empty subset of A which is closed under the operation $\odot$. Note that any maximal filter is a prime filter.

PROPOSITION 1.7. *Assume that A is an MV-algebra.*

(1) *If F is a filter of A then the relation Θ_F defined by*

$$(x, y) \in \Theta_F \quad \text{if} \quad \neg d(x, y) \in F$$

is a congruence of A.

(2) *If Θ is a congruence of A, then $1/\Theta$ is a filter of A.*

(3) *The maps defined in the two previous items are two inverse isomorphisms between the lattice of filters of A and the lattice of congruences of A.*

The previous proposition allows us to use the well established notations : we denote by A/F the algebra A/Θ_F when F is a filter of A.

The following result justifies the terminology since it is the equivalent for the variety of MV-algebras of the STONE extension theorem for boolean algebras.

THEOREM 1.8. *Assume that A is an MV-algebra, that F is a filter of A and that a is an element of A that does not belong to F. Then there is a prime filter F' of A that contains F but not a.*

Unfortunately, unlike boolean algebras, in an MV-algebra, maximal filters do not coincide with prime filters. Nevertheless, they coincide in any finitely generated variety.

PROPOSITION 1.9. *Assume that A is an MV-algebra and that F is a filter of A.*

(1) *The algebra A/F is an MV-chain if and only if F is a prime filter of A.*
(2) *The algebra A/F is simple if and only if F is a maximal filter of A.*
(3) *The algebra A is a subdirect-product of MV-chains.*
(4) *The algebra A is simple if and only if there is a unique embedding $A \hookrightarrow [0, 1]$.*

2. More about finitely generated varieties

We have already stated that the finitely generated varieties of MV-algebras are exactly the varieties generated by a finite number of finite MV-chains. Among them, the varieties $\mathbb{HSP}(Ł_n)$ generated by a single MV-chain are even more interesting since any finitely generated variety of MV-algebras is a subvariety of $\mathbb{HSP}(Ł_n)$ for a positive integer n. Moreover, the variety $\mathbb{HSP}(Ł_n)$, that we denote by $\mathcal{MV}_n$, is the algebraic countepart of Łukasiewicz $n+1$-valued logic.

Proposition 1.10. *The variety $\mathcal{MV}_n$ is obtained from the variety $\mathcal{MV}$ by adding to the axiomatisation of $\mathcal{MV}$ the equations $(px^{p-1})^{n+1} \leftrightarrow (n+1)x^p$ for any prime $p < n$ that does not divide n and the formula $(n+1)x \leftrightarrow nx$.*

Algebras of $\mathcal{MV}_n$ can be represented as a boolean products of subalgebras of $Ł_n$.

Definition 1.11. Assume that A is a member of $\mathcal{MV}_n$. We denote by $\mathcal{MV}(A, Ł_n)$ the set of MV-homomorphisms from A to $Ł_n$. If a belongs to A and i belongs to $\{0, \ldots, n\}$, we denote by $[a : \frac{i}{n}]$ the set $\{u \in \mathcal{MV}(A, Ł_n) \mid u(a) = \frac{i}{n}\}$.

We equip the set $\mathcal{MV}(A, Ł_n)$ with the topology that has $\{[a : \frac{i}{n}] : a \in A, i \in \{0, \ldots, n\}\}$ as a subbase.

Proposition 1.12. *Assume that A is an MV_n-algebra.*

(1) *the topological space $\mathcal{MV}(A, Ł_n)$ is a boolean space (*i.e.*, a compact,* Hausdorff *and zero-dimensional space),*
(2) *the evaluation map*

$$e_A : A \to \prod_{u \in \mathcal{MV}(A, Ł_n)} : a \mapsto (u(a))_{u \in \mathcal{MV}(A, Ł_n)}$$

provides a representation of A as a boolean product of its simple quotients.

3. More about MV-terms

Let us recall some folklore results about the construction of some MV-terms. The idea of using "special" MV-terms to develop many-valued modal systems can be traced back to [**48**]. In the sequel of the dissertation, we denote by $\mathbb{N}_0$ the set of the positive integers and by $\mathbb{N}$ the set of the strictly positive integers (*i.e.*, we have $0 \in \mathbb{N}_0$ but $0 \notin \mathbb{N}$).

Definition 1.13. The set $\mathbb{D}$ of *dyadic numbers* is the set of the rational numbers that can be written as a finite sum of power of 2.

If a is a number of $[0, 1]$, a *dyadic decomposition of* a is a sequence $a^* = (a_i)_{i \in \mathbb{N}}$ of elements of $\{0, 1\}$ such that $a = \sum_{i=1}^{\infty} a_i 2^{-i}$. We denote by a_i^* the i^{th} element of any sequence (of length greater than i) a^*.

If a is a dyadic number of $[0, 1]$, then a admits a unique finite dyadic decomposition, called *the dyadic decomposition of* a.

If a^* is a dyadic decomposition of a real a and if k is a positive integer then we denote by $\ulcorner a^* \urcorner_k$ the finite sequence $(a_1, \ldots, a_k)$ defined by the first k elements of a^*.

We temporarily denote by $f_0(x)$ and $f_1(x)$ the terms $x \oplus x$ and $x \odot x$ respectively, and by $T_{\mathbb{D}}$ the clone generated by $f_0(x)$ and $f_1(x)$. Further in the dissertation, we denote by $\tau_{\oplus}$ the term f_0 and by $\tau_{\odot}$ the term f_1.

We also denote by $g_{\bullet}$ the mapping between the set of finite sequences of elements of $\{0,1\}$ (and thus of dyadic numbers in $[0,1]$) and $T_{\mathbb{D}}$ defined by:

$$g_{(a_1,\ldots,a_k)} = f_{a_k} \circ \cdots \circ f_{a_1}$$

for any finite sequence $(a_1,\ldots,a_k)$ of elements of $\{0,1\}$. If $a = \sum_{i=1}^k a_i 2^{-i}$, we sometimes write g_a instead of $g_{(a_1,\ldots,a_k)}$.

The terms in $T_{\mathbb{D}}$ are heavily used in the sequel of this dissertation. We therefore give some details about these terms.

LEMMA 1.14. *If $a^* = (a_i)_{i\in\mathbb{N}}$ and $x^* = (x_i)_{i\in\mathbb{N}}$ are dyadic decompositions of two elements of $[0,1]$, then, for any positive integer k,*

$$g_{\ulcorner a^* \urcorner_k}(x) = \begin{cases} 1 & \text{if } x > \sum_{i=1}^k a_k 2^{-k} + 2^{-k} \\ 0 & \text{if } x < \sum_{i=1}^k a_k 2^{-k} \\ \sum_{i=1}^\infty x_{i+k} 2^i & \text{if } \sum_{i=1}^k a_k 2^{-k} \leq x \leq \sum_{i=1}^k a_k 2^{-k} + 2^{-k}. \end{cases}$$

PROOF. We proceed by induction on k. If $k=1$, the result is clear. Let us assume that the decomposition of $g_{\ulcorner a^* \urcorner_l}(x)$ is obtained for any $l<k$ and let us obtain the decomposition of $g_{\ulcorner a^* \urcorner_k}(x)$. By definition,

$$g_{\ulcorner a^* \urcorner_k}(x) = f_{a_k}(g_{\ulcorner a^* \urcorner_{k-1}}(x)).$$

Thus, if $x > \sum_{i=1}^{k-1} a_i 2^{-i} + 2^{-(k-1)}$, we obtain by induction hypothesis that $g_{\ulcorner a^* \urcorner_k}(x) = 1$ and that $g_{\ulcorner a^* \urcorner_k}(x) = 0$ if $x < \sum_{i=1}^{k-1} a_i$. Let us now assume that

$$\sum_{i=1}^{k-1} a_i \leq x \leq \sum_{i=1}^{k-1} a_i 2^{-i} + 2^{-(k-1)}.$$

If $a_k = 0$, we conclude successively that

$$\begin{aligned} g_{\ulcorner a^* \urcorner_k}(x) &= f_0(\sum_{i=1}^\infty x_{i+(k-1)} 2^{-i}) \\ &= \begin{cases} 1 & \text{if } x_k = 1 \\ 2.\sum_{i=1}^\infty x_{i+(k-1)} 2^{-i} & \text{if } x_k = 0. \end{cases} \end{aligned}$$

Thus, we have obtained that if

$$x \in]\sum_{i=1}^{k-1} a_i 2^{-i} + 2^{-(k-1)}, 1] \cup [\sum_{i=1}^{k-1} a_i 2^{-i} + 2^{-k}, \sum_{i=1}^{k-1} a_i 2^{-i} + 2^{-(k-1)}] =]\sum_{i=1}^{k} a_i 2^{-i} + 2^{-k}, 1]$$

then $g_{\ulcorner a^* \urcorner_k}(x) = 1$, that if

$$x \in [0, \sum_{i=1}^k a_i 2^{-i}[$$

then $g_{\ulcorner a^* \urcorner_k}(x) = 0$, and eventually that if

$$x \in [\sum_{i=1}^k a_i 2^{-i}, \sum_{i=1}^k a_i 2^{-i} + 2^{-k}]$$

then $g_{\ulcorner a^* \urcorner_k}(x) = \sum_{i=1}^\infty x_{i+k} 2^{-i}$, which is the desired result when $a_k = 0$. We proceed in a similar way when $a_k = 1$. $\square$

COROLLARY 1.15. *If r and s are two dyadic numbers of $[0,1]$ and if n is a positive integer,*

(1) *there is a term τ_r in $T_{\mathbb{D}}$ such that $\tau_r(x) = 1$ if and only if $x \geq r$;*
(2) *if $s < r$, then there is a term $\tau_{s;r}$ in $T_{\mathbb{D}}$ such that*

$$\tau_{s;r}(x) = \begin{cases} 1 & \text{if } x \geq r \\ 0 & \text{if } x \leq s \end{cases} \qquad \text{and} \qquad \tau_{s;r}(x) < 1 \text{ if } x \in]s, r[$$

(3) *if $i \in \{1, \ldots, n\}$, then there is a term $\tau_{i/n}$ in $T_{\mathbb{D}}$ such that if x belongs to $Ł_n$, then*

$$\tau_{i/n}(x) = \begin{cases} 0 & \text{if } x < \frac{i}{n} \\ 1 & \text{if } x \geq \frac{i}{n}. \end{cases}$$

PROOF. (1) If $r = \sum_{i=1}^{k} r_i 2^{-i}$ with $r_k = 1$, then we can set $\tau_r = g_{(r_1,\ldots,r_{k-1},0)}$.
(2) If $s = \sum_{i=1}^{k} s_i 2^{-i}$ with $s_k = 1$, then the desired term can be obtained by considering

$$g_{(0,\ldots,0)} \circ g_{(s_1,\ldots,s_k)}$$

for a suitable finite sequence $(0, \ldots, 0)$ of 0.
(3) The term $\tau_{i/n}$ can be obtained if we apply (2) with $s < r$ in $[\frac{i-1}{n}, \frac{i}{n}]$. □

We can use the terms $\tau_{i/n}$ to compute a term that recognize elements of $Ł_n$ that belongs to $Ł_m$ (where m is a divisor of n).

DEFINITION 1.16. Assume that m is in $\mathrm{div}(n)$. We denote by $\tau_{Ł_m}$ the term

$$\tau_{Ł_m}(x) = \bigvee_{i \in \{0,\ldots,m\}} \tau_{i(n/m)/n}(x) \wedge \neg\tau_{i(n/m)+1/n}(x)$$

We obviously obtain that

$$\tau_{Ł_m}(x) = \begin{cases} 1 \text{ if } x \in Ł_m, \\ 0 \text{ otherwise.} \end{cases}$$

for any x in $Ł_n$.

4. Complete MV-algebras

Complex algebras that appear in the sequel of the dissertation are complete and completely distributive. We recall here a few results about these algebras.

LEMMA 1.17. *If A is a complete MV-algebra, then $\mathfrak{B}(A)$ is a complete boolean algebra. Moreover, if A is completly distributive, so is $\mathfrak{B}(A)$.*

LEMMA 1.18. *If A is a complete MV-algebra and if $\{b_j \mid j \in J\}$ is a subset of $\mathfrak{B}(A)$ such that $b_j \wedge b_k = 0$ for any $j \neq k$ in J and $\bigvee\{b_j \mid j \in J\} = 1$, then A is isomorphic to $\prod_{j \in J}(b_j]$.*

LEMMA 1.19. *If A is a complete MV-algebra and a is an atom of $\mathfrak{B}(A)$ then,*

(1) *if there is a atom of A above a, then the MV-algebra $(a]$ is a finite MV-chain;*
(2) *if there is no dual atom above a, then the MV-algebra $(a]$ is isomorphic to $[0,1]$.*

Thus, the only totally ordered complete MV-algebras are the algebras $[0,1]$ and $Ł_n$ for $n \in \mathbb{N}$.

We now turn to complete and completely distributive MV-algebras. Since any complete and completely distributive boolean algebra is atomic and is isomorphic to the powerset algebra of a set, we can conclude from Lemma 1.18 that if A is a complete and completely distributive MV-algebra, then A is isomorphic to $\prod\{(a] \mid a \in \mathsf{Atom}(\mathfrak{B}(A))\}$. Moreover for any a in $\mathsf{Atom}(A)$, the algebra $(A]$ is isomorphic to $A/(\neg a]$, *i.e.*, $(\neg a]$ is a maximal filter of A. We turn the preceding lines into the following result.

LEMMA 1.20. *An MV-algebra A is complete and completely distributive if and only if it is isomorphic to a direct product of finite MV-chains and isomorphic copies of* $[0,1]$. *More precisely, the algebra A is complete and completely distributive if and only if the map*

$$h_A : A \to \prod_{a \in \mathsf{Atom}(\mathfrak{B}(A))} (a] : x \mapsto (x \wedge a)_{a \in \mathsf{Atom}(\mathfrak{B}(A))}$$

is an MV-isomorphism.

If A is a complete and completely distributive MV-algebra, we will often work modulo the isomorphism h_A of Lemma 1.20. Note that in $h_A(A)$ the $\vee$-irreducible elements are simply the elements $(x_a)_{a \in \mathsf{Atom}(\mathfrak{B}(A))}$ for which there is a b in $\mathsf{Atom}(\mathfrak{B}(A))$ such that $x_a = 0$ if $a \neq b$.

CHAPTER 2

Models and structures

1. Language and models

The standard definition a KRIPKE model can be adapted to a many-valued (ŁUKASIEWICZ) realm in a straightforward way. The basic idea is to set the truth value of any propositional variable in each world in a complete MV-algebra A. We temporally consider models that are valued in any complete MV-algebra, but from subsection 1.2 to the end of the dissertation, we will only consider $[0,1]$-valued models. KRIPKE models for many-valued modal logics have already been considered (see, *e.g.*, [**15**], [**13**], [**14**]). Our approach is a generalization of the approach of [**48**].

Let us first introduce the languages of the formulas that are interpreted in these models.

DEFINITION 2.1. A *language* is a set $\mathcal{L}$ of symbols, called *connectives* together with a map $n. : \mathcal{L} \to \mathbb{N} : f \mapsto k_f$ where k_f is the *arity* of f. Connectives with arity 0 are called *constants.* If $\mathcal{L}$ is a similarity type, the *$\mathcal{L}$-formulas over the set of variables* Prop are defined inductively by the following rules:

- every variable p of Prop is an $\mathcal{L}$-formula,
- if f is a connective of $\mathcal{L}$ with arity k and if $\phi_1, \ldots, \phi_k$ are $\mathcal{L}$-formulas then $f(\phi_1, \ldots, \phi_k)$ is an $\mathcal{L}$-formula.

In the sequel, if not stated otherwise, we assume that Prop is a enumerable set of propositional variables and that $\mathcal{L}$ is a language $\mathcal{L} = \{\to, \neg, 0, 1\} \cup \{\nabla_i \mid i \in I\}$ such that the connective $\to$ is of arity 2, the connective $\neg$ is unary, and 0 and 1 are constants. We denote by $\mathsf{Form}_{\mathcal{L}}$ the set of the $\mathcal{L}$-formulas over the set of variables Prop. Note that, by using the following standard abbreviations:

$$\begin{array}{rr} \phi \oplus \psi := \neg\psi \to \phi & \phi \odot \psi := \neg(\neg\phi \oplus \neg\psi) \\ \phi \vee \psi := (\phi \to \psi) \to \psi & \phi \wedge \psi := \neg(\neg\phi \vee \neg\psi) \\ \phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi) & \Delta_i(\phi_1, \ldots, \phi_{k_i}) := \neg\nabla_i(\neg\phi_1, \ldots, \neg\phi_{k_i}), \end{array}$$

we can usually consider that the connectives $\oplus$, $\odot$, $\vee$, $\wedge$, $\leftrightarrow$ and Δ_i (with $i \in I$) belong to $\mathcal{L}$. Indeed, the problem of defining a set of primitive connectives does not play any key role in our developments but in Chapter 4 in which we are more careful about the subject.

DEFINITION 2.2. The elements of $\{\nabla_i \mid i \in I\}$ are called *dual MV-modalities* or *dual modalities.* The dual Δ_i of any dual MV-modality ∇_i is named an *MV-modality* or a *modality.* The arity of ∇_i is denoted by k_i. For any j in $\{1, \ldots, k_i\}$ we use the following abbreviation

$$\nabla^{(j)}(p) = \nabla(0, \ldots, 0, p, 0, \ldots, 0)$$

where p is the j^{th} element in $(0, \ldots, 0, p, 0, \ldots, 0)$.

We call any unary dual MV-modality a *box* and its dual a *diamond*. We denote boxes by symbols like $\Box, [a]$ and diamonds by $\Diamond, \langle a \rangle$ etc.

We decide to call *dual MV-modalities* (and not simply *MV-modalities*) the primitive connectives ∇_i in order to reconnect with classical definitions of modal logics. This will appear clearly in the sequel. Moreover, working with modalities or dual modalities as primitive connectives is a matter of taste, culture and opportunism.

DEFINITION 2.3. If A is a complete MV-algebra, an *A-valued* KRIPKE *$\mathcal{L}$-model*

$$\mathcal{M} = \langle W, \{R_i \mid i \in I\}, \mathrm{Val}\rangle$$

is given by a non empty set W (the elements of W are often called *worlds*, *points* or *states*, and W is often called the *universe* or the *carrier* of $\mathcal{M}$), an *accessibility relation* $R_i \subseteq W^{k_i+1}$ for any modality ∇_i of arity k_i in $\{\nabla_i \mid i \in I\}$ and a map $\mathrm{Val} : \mathsf{Prop} \times W \to A$, called the *valuation of the model.*

When A or $\mathcal{L}$ is clearly determined by the context, we often abbreviate "A-valued KRIPKE $\mathcal{L}$-model" by *$\mathcal{L}$-model* or *A-valued* KRIPKE *model* or *A-valued model* or *model.* The class that contains every A-valued $\mathcal{L}$-model for every complete MV-algebra A is the class of the *many-valued $\mathcal{L}$-models* or *many-valued models.*

Sometimes, when we feel the need to be more precise, we denote by $\mathrm{Val}_{\mathcal{M}}$ and $R_i^{\mathcal{M}}$ with (i in I) the valuation map of $\mathcal{M}$ and the accessibility relations of $\mathcal{M}$ respectively.

If R is a k+1-ary relation on W and if u belongs to W, we denote by $R(u)$ or by Ru the *set of successors* $\{(v_1, \dots, v_k) \mid (u, v_1, \dots, v_k) \in R\}$ of u.

We extend the valuation to formulas in the natural way. In the sequel, we use the same notations $\to$ and $\neg$ to denote the previously defined connectives but also the corresponding operations $\to^A$ and $\neg^A$ on an MV-algebra A (recall for example that $\to^{[0,1]}$: $[0,1]^2 \to [0,1] : (x, y) \mapsto \min(1, 1 - x + y)$ and $\neg^{[0,1]} : [0,1] \to [0,1] : x \mapsto 1 - x$). The following lemma is used a a definition.

LEMMA 2.4. *If $\mathcal{M} = \langle W, R, \mathrm{Val}\rangle$ is a many-valued $\mathcal{L}$-model, then there is a unique extension $\mathrm{Val}' : \mathsf{Form} \to [0,1]$ of the map Val that satisfies the two following conditions:*

- $\mathrm{Val}'(w, \phi \to \psi) = \mathrm{Val}'(w, \phi) \to \mathrm{Val}'(w, \psi)$ *and* $\mathrm{Val}'(w, \neg\psi) = \neg\mathrm{Val}'(w, \psi)$ *for any w in W and any ϕ and ψ in* Form*,*
- $\mathrm{Val}'(w, \nabla_i(\phi_1, \dots, \phi_{k_i})) = \bigwedge\{\mathrm{Val}'(v_1, \phi_1) \vee \dots \vee \mathrm{Val}'(v_{k_i}, \phi_{k_i}) \mid (w, v_1, \dots, v_{k_i}) \in R_i\}$ *for any ∇_i of arity k_i in $\{\nabla_i \mid i \in I\}$;*

Thanks to this unicity property, we simply denote the map Val' *by* Val.

The previous Lemma explains why we have restricted our definition of A-valued $\mathcal{L}$-models to MV-algebras A that are complete (we really need to be able to compute infinite meets).

We may deduce the following useful and easy result that details how the valuation maps act on the dual Δ_i of a modality ∇_i.

LEMMA 2.5. *If $\mathcal{M}$ is a $\mathcal{L}$-model, then for any w in W, any i in I and any $\mathcal{L}$-formulas $\phi_1, \dots, \phi_{k_i}$,*

$$\mathrm{Val}(w, \Delta_i(\phi_1, \dots, \phi_{k_i})) = \bigvee\{\mathrm{Val}(v_1, \phi_1) \wedge \dots \wedge \mathrm{Val}(v_{k_i}, \phi_{k_i}) \mid (w, v_1, \dots, v_{k_i}) \in R_i\}.$$

The definition of the satisfaction relation in a model $\mathcal{M}$ follows the classical one.

DEFINITION 2.6. If w is a world of an $\mathcal{L}$-model $\mathcal{M}$ and if ϕ is a $\mathcal{L}$-formula such that $\mathrm{Val}_{\mathcal{M}}(w,\phi)=1$, we say that ϕ *is true in* w and write $\mathcal{M},w\models\phi$. When $\mathcal{M},w\models\phi$ for any world w of $\mathcal{M}$, we say that ϕ *is true in* $\mathcal{M}$ and write $\mathcal{M}\models\phi$. A formula that is true in any model of a class M of models is called an M*-tautology* (or simply a *tautology* if M is clearly determined by the context or if it is a tautology in any many-valued model).

We illustrate this definition.

PROPOSITION 2.7. *It τ is a unary formula constructed only with the connectives $\oplus$ and $\odot$ then the formulas*

(1) $\nabla^{(i)}(p\to q)\to(\nabla^{(i)}(p)\to\nabla^{(i)}(q))$,
(2) $\nabla^{(i)}(p\wedge q)\leftrightarrow(\nabla^{(i)}(p)\wedge\nabla^{(i)}(q))$,
(3) $\tau(\nabla(p_1,\ldots,p_k))\leftrightarrow\nabla(\tau(p_1),\ldots,\tau(p_k))$

are tautologies for any k-ary dual MV-modality ∇ and any i in $\{1,\ldots,k\}$.

1.1. Models valued in complete and completely distributive MV-algebras. We have seen in Lemma 1.20 that any complete and completely distributive MV-algebra A is canonically isomorphic to a product of finite MV-chains and copies of $[0,1]$. From that result, we are going to deduce that, in some way, any problem about validity of formulas in A-valued models where A is complete and completely distributive, is equivalent to a problem about validity of the same formulas in $[0,1]$-valued models.

Recall for such an algebra A, the map

$$h_A : A\to\prod_{p\in\mathsf{Atom}(\mathfrak{B}(A))}(a] : x\mapsto(x\wedge p)_{p\in\mathsf{Atom}(\mathfrak{B}(A))}$$

is an isomorphism. Moreover, for any p in $\mathsf{Atom}(\mathfrak{B}(A))$, the algebra $(p]$ is a finite MV-chain or a copy of $[0,1]$ and we denote by π_p the map $i'_p\circ h_p$ where

$$h_p : A\twoheadrightarrow(p] : x\mapsto x\wedge p$$

and i'_p is the unique embedding of $(p]$ into $[0,1]$.

DEFINITION 2.8. Assume that A is a complete and completely distributive MV-algebra and that $\mathcal{M}=\langle W,\{R_i\mid i\in I\},\mathrm{Val}\rangle$ is an A-valued $\mathcal{L}$-model. The *unraveled* $[0,1]$*-valued model associated to* $\mathcal{M}$ is the model $\mathcal{M}^u=\langle W^u,\{R_i^u\mid i\in I\},\mathrm{Val}^u\rangle$ where

- the universe W^u of $\mathcal{M}^u$ is equal to $\bigcup\{\{w\}\times\mathsf{Atom}(\mathfrak{B}(A))\mid w\in W\}$,
- for any i in I the relation R_i^u contains $((u,p),(v_1,q_1),\ldots,(v_{k_i},q_{k_i}))$ if $p=q_1=\cdots=q_{k_i}$ and $(u,v_1,\ldots,v_{k_i})$ belongs to R_i,
- for any propositional variable r of Prop and any world (v,p) of W^u, the truth value $\mathrm{Val}^u((v,p),r)$ of r in (v,p) is equal to $\pi_p(\mathrm{Val}(v,r))$.

Actually, the definition of Val^u extends nicely to formulas.

LEMMA 2.9. *Assume that A is a complete and completely distributive MV-algebra and that $\mathcal{M}=\langle W,\{R_i\mid i\in I\},\mathrm{Val}\rangle$ is an A-valued model with unraveled $[0,1]$-valued model $\mathcal{M}^u$. Then,*

$$\mathrm{Val}^u((v,p),\phi)=\pi_p(\mathrm{Val}(v,\phi))$$

for any $\mathcal{L}$-formula ϕ and any world (v,p) of W^u.

PROOF. The proof is done by induction on the number of connectives in ϕ. The induction step $\phi = \nabla(\psi_1, \dots, \psi_k)$ is the only step that deserves some attention. But the proof is an exercise if we note that π_p is a complete homomorphism for any p in $\mathsf{Atom}(\mathfrak{B}(A))$. □

COROLLARY 2.10. *Assume that A is a complete and completely distributive MV-algebra.*

(1) *A formula ϕ is satisfiable or refutable in the class of A-valued models if and only if it is respectively satisfiable or refutable in the class of $[0,1]$-valued model.*
(2) *If M is a class of A-valued models and if $\mathbf{L}$ denotes $\{\phi \in \mathsf{Form}_{\mathcal{L}} \mid \forall \mathcal{M} \in M, \mathcal{M} \models \phi\}$, then there is a class M' of $[0,1]$-valued models such that $\mathbf{L} = \{\phi \in \mathsf{Form}_{\mathcal{L}} \mid \forall \mathcal{M} \in M', \mathcal{M} \models \phi\}$*

Since every complete MV-algebra A is semi-simple, *i.e.*, is canonically embeddable in a power of $[0,1]$ (see [**39**]), we could hope to extend these results to the class of A-valued models for any complete MV-algebra A. But in trying to generalize these results we come up against the fact that in the subdirect representation

$$i_A : A \hookrightarrow \prod_{F \in \mathrm{Max}(A)} A/F$$

of A by its simple quotients the maps $p_F \circ i_A$ (where p_F denotes the projection map of $\prod\{A/F \mid F \in \mathrm{Max}(A)\}$ onto A/F) is not a complete homomorphism.

From now on, we only consider A-valued models where A is a complete subalgebra of $[0,1]$.

1.2. Models for MPDL. We introduce the languages and the models for a many-valued generalization of propositional dynamic logic with tests. (Propositional) dynamic logic (PDL) is a modal logic of programs introduced by in [**30**]. We refer any reader that is not already acquainted with PDL to [**31**].

To define formulas of MPDL, we need, besides a (countable) set of propositional variables Prop (the elements of Prop are denoted by $p, q, \dots, p_1, p_2, \dots$), a set Π_0 of atomic programs (that are denoted by $a, b, \dots, a_1, b_1, \dots$). The set Π of programs and Form of well formed formulas are defined by mutual induction according to the following rules:

- any atomic program a of Π_0 is a program of Π and any propositional variable p of Prop is a formula of Form;
- if α and β are in Π than so are $\alpha;\beta$, $\alpha \cup \beta$ and α^*;
- if ϕ and ψ are in Form then so are $\phi \to \psi$ and $\neg\psi$;
- if α belongs to Π and ϕ belongs to Form then $[\alpha]\phi$ is a formula of Form and $\psi?$ is a program of Π.

Thus, we have an infinite set of modalities in the language of MPDL since it contains a modality $[\alpha]$ for any program α of Π. For the sake of readability, we consider that Π_0 and Prop are defined once for all. A KRIPKE model for MPDL is thus defined with an infinite set of binary relations : each program α of Π has its corresponding relation R_α on the model. In order to model the program operators ; , $*$ and $\cup$, we define the relation associated to a program α from the relations of the atomic programs that appear in α.

DEFINITION 2.11. If A is a complete subalgebra of $[0,1]$, an *A-valued* KRIPKE *model for MPDL* (or simply an A-valued KRIPKE *model*) $\mathcal{M} = \langle W, R_{\boldsymbol{.}}, \mathrm{Val}\rangle$ is given by a non empty set W, a map $R_{\boldsymbol{.}} : \Pi_0 \to 2^{W \times W}$ that assigns a binary relation R_a to every atomic program a of

Π_0 and a map $\mathrm{Val} : W \times \mathsf{Prop} \to A$ that assigns a truth value to each propositional variable p of Prop in any world w of W.

The maps Val and $R_{\bullet}$ are extended to formulas and programs respectively by mutual induction according to the following rules.

- for any programs α and β, we set $R_{\alpha;\beta} = R_\alpha \circ R_\beta$, and $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$;
- for any formula ψ, we define $R_{\psi?}$ as the relation $\{(u,u) \mid \mathrm{Val}(u,\psi) = 1\}$;
- for any program α, the relation R_{α^*} is the transitive and reflexive closure of R_α, that is $R_{\alpha^*} = \bigcup_{n\in\omega} R_{\alpha^n}$ where α^n denotes the composition under ; of n factors α.
- if ϕ and ψ are two formulas and w belongs to W then $\mathrm{Val}(w, \phi \to \psi) = \mathrm{Val}(w,\phi) \to \mathrm{Val}(w,\psi)$ and $\mathrm{Val}(w, \neg\psi) = \neg\mathrm{Val}(w,\psi)$.
- if ψ is a formula, if α is a program and if w belongs to W then $\mathrm{Val}(w, [\alpha]\psi) = \bigwedge_{v\in R_\alpha w} \mathrm{Val}(v,\psi)$.

Our intentions are clear : we want to interpret the operator ; as the concatenation operator, the operator $\cup$ as the alternative operator and the operator $*$ as the KLEENE operator. Hence, if α and β are programs, the connective $[\alpha]$ is read "after any execution of α", the connective $[\alpha\cup\beta]$ is read "after any execution of α or β", the connective $[\alpha;\beta]$ is read "after any consecutive execution of α and β" and $[\alpha^*]$ is read "after any finite number of executions of α".

It is now time to produce a few examples of tautologies.

PROPOSITION 2.12. *Assume that A is a complete subalgebra of $[0,1]$. The following formulas are true in any A-valued* KRIPKE *model :*

(1) $[\alpha \cup \beta]p \leftrightarrow [\alpha]p \wedge [\beta]p$.
(2) $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$.
(3) $\langle\alpha \cup \beta\rangle p \leftrightarrow \langle\alpha\rangle p \vee \langle\beta\rangle p$.
(4) $\langle\alpha;\beta\rangle p \leftrightarrow \langle\alpha\rangle\langle\beta\rangle p$.
(5) $[\alpha^*]p \to p$.
(6) $p \to \langle\alpha^*\rangle p$.
(7) $[\alpha^*]p \to [\alpha]p$.
(8) $\langle\alpha\rangle p \to \langle\alpha^*\rangle p$.
(9) $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$.
(10) $\langle\alpha^*\rangle p \leftrightarrow (p \vee \langle\alpha\rangle\langle\alpha^*\rangle p)$.

If $A = \text{Ł}_n$ then the formula $p \wedge ([\alpha^*](p \to [\alpha]p)^n) \to [\alpha^*]p$ is a tautology.

1.3. An example: the RÉNYI-ULAM Game. We can use the previously defined models to provide a framework for an interpretation of the famous RÉNYI-ULAM game. ULAM's formulation of the game in [**54**], which was previously and independently introduced by RÉNYI, is the following:

> Someone thinks of a number between one and one million (which is just less than 2^{20}). Another person is allowed to ask up to twenty questions, to each of which the first person is supposed to answer only yes or no. Obviously the number can be guessed by asking first: is the number in the first half-million? and again reduce the reservoir of numbers in the next question by one-half, and so-on. Finally, the number is obtained in less than $\log_2 1000000$. Now,

suppose that one were allowed to lie once or twice, then how many questions would one need to get the right answer?

Lots of researchers (mainly computer scientists) have focused their attention on that game since the publication of ULAM's book [**54**]. The success of the game is due to its connection with the theory of error-correcting codes with feedbacks in a noisy channel and the complexity of the problem of finding an optimal strategy for the game. We refer to [**49**] for an overview of the literature about the RÉNYI-ULAM game.

The game has also been considered by many-valued logicians as a way to give a concrete interpretation of ŁUKASIEWICZ finitely-valued calculi and their associated algebras (see [**45**]). Up to now, mathematician have modeled the game in a simple way by coding algebraically *questions* and *answers*. We start by introducing this interpretation, which is due to MUNDICI, and then add to it a dynamic layer in order to model the interactions between the two gamers.

1.3.1. *Algebraic approach of the states of knowledge.* We call the first gamer (the one who chooses a number and can lie) Pinocchio, and the second gamer Geppetto. Let us denote by M the *search space*, *i.e.*, the finite set of integers (or whatever) in which Pinocchio can pick up his number. Let us also assume that Pinocchio can lie $n-1$ times (where $n \geq 1$).

We first have to determine a way to algebraicly encode the information defined by Pinocchio's answers, *i.e.*, to model Geppetto's state of knowledge of the game after each of Pinocchio's answers. This can be done by considering at step i of the game the map

$$r_i : M \to \{0, 1, \dots, n\}$$

where $r_i(m)$ is defined as the number of the i previous answers that refute the element m of M as Pinocchio's number. Indeed, once $r(m) = n$, since Pinocchio is allowed to lie $n-1$ times, Geppetto can safely conclude that m is not the "right" number. Hence, the game ends once Geppetto encodes its knowledge by a map r which is null except for one m in M, which is the searched number. In such a final state, Geppeto can even determine the number of lies of Pinocchio in the game: this numer is equal to $n - r(m)$.

In order to introduce MV-algebras in the interpretation of the game, we will consider a equivalent representation of Geppetto's states of knowledge. The approach we now consider was introduced in [**45**].

DEFINITION 2.13. A *state of knowledge* is a map $f : M \to Ł_n$. The *state of knowledge* f *at some step of the game* is the state of knowledge f defined by $f(m) = 1 - \frac{r(m)}{n}$ where $r(m)$ denotes for any m in M the number of Pinocchio's answers that refute m as the searched number.

Hence, if f is a state of knowledge at some step of the game, the number $f(m)$ can be viewed for any m in M as the relative distance between m and the set of the elements of M that can be safely discarded as unappropriate.

1.3.2. *Questions and answers.* We are now concerned by the modifications that have to be taken into account in the states of knowledge between two steps of the games, *i.e.*, after one of Pinocchio's answer. First note that any question that Geppetto can ask is equivalent to a question of the form "Does the searched number belong to Q?" for a subset Q of the search space M. Hence, in the sequel, we will denote any question by its associated subset Q of M.

Let us assume that Geppetto has reached the state of knowledge f and that he asks question Q. What is the state of knowledge f' of the game after Pinocchio's answer? If Pinocchio answers positively ("Yes, the number belongs to Q") then Gepetto increments $r(m)$ by one (if necessary) for any m in $M \setminus Q$ since a positive answer to Q is equivalent to a negative answer to $M \setminus Q$, *i.e.*,

$$f' : M \to Ł_n = m \mapsto \begin{cases} f(m) & \text{if } m \in Q \\ \max\{f(m) - \frac{1}{m}, 0\} & \text{if } m \in M \setminus Q. \end{cases}$$

On the contrary, if Pinocchio answers negatively to Q, then Gepetto increments $r(m)$ by one (if necessary) for any m in Q, *i.e.*,

$$f' : M \to Ł_n = m \mapsto \begin{cases} f(m) & \text{if } m \in M \setminus Q \\ \max\{f(m) - \frac{1}{m}, 0\} & \text{if } m \in Q. \end{cases}$$

This line of argument justifies the following definition.

Definition 2.14. If Q is a subset of M, the *positive answer* to Q is the map

$$f_Q : M \to \{\frac{n-1}{n}, 1\} : m \mapsto \begin{cases} 1 & \text{if } m \in Q \\ \frac{n-1}{n} & \text{if } m \in M \setminus Q. \end{cases}$$

The *negative answer* to Q is the positive answer $f_{M \setminus Q}$ to $M \setminus Q$.

We can thus encode algebraically any of Pinocchio's answers.

Lemma 2.15. *Assume that Geppetto has reached the state of knowledge f and that he asks question Q. After Pinocchio's answer to Q, the stage of knowledge f' of the game is $f \circ f_Q$ if Pinocchio's answer is positive and $f \circ f_{M \setminus Q}$ if it is negative.*

1.3.3. *A dynamic layer.* Roughly speaking, we have modeled the game in a static way. There is no possibility yet to model all the possible interactions between the gamers. We now add our touch to this interpretation of the game by encoding every possible run of the game in an $Ł_n$-valued Kripke model for MPDL. The idea of the construction of the model is clear: the universe of the model is the set $Ł_n^M$ of the states of knowledge, the set of atomic programs Π_0 is the set of questions 2^M and the set of propositional variables $\{p_m \mid m \in M\}$ that are relevant to the problem is made of a variable p_m for any m in M that can be read as "m is far from the set of rejected element" or "the relative distance between m and the set of rejected elements is".

Definition 2.16. The model of the Rényi - Ulam game with search space M and $n-1$ lies is the $Ł_n$-valued Kripke model for MPDL $\mathcal{M} = \langle Ł_n^M, R_., \mathrm{Val}\rangle$ where

- the set of atomic programs is $\Pi_0 = 2^M$,
- for any Q in 2^M, the relation R_Q contains (f, f') if $f' = f \circ f_Q$ or if $f' = f \circ f_{M \setminus Q}$,
- for any m in M and any f in $Ł_n^M$, we set $\mathrm{Val}(f, p_m) = f(m)$.

This model provides a way to interpret any run of the game as a path from the initial state $f : m \mapsto 1$ to any final state. Examples of formulas that are true in the model are

$(\neg p_m)^n \to [Q]\neg p_m$ and $\tau_{\frac{i}{n}}(p_m) \to [Q; M \setminus Q]\tau_{\frac{i-2}{n}}(p_m)$ (if we agree that $\tau_{\frac{i-2}{n}}(p_m) = 1$ if $i-2 \leq 0$).

What are the formulas that are true in any model for MPDL_n? That is, what are the properties that are shared in any model of programs? This question is answered in subsection 4.3 of chapter 3 in which we develop an axiomatic system that generates any MPDL_n-tautology.

2. Structures, frames and frame constructions

If we remove the contingent information provided by the valuation map in the definition of a many-valued Kripke model, the object we obtain is a very simple structure : it is a relational structure that has a $k+1$-ary relation R for any k-ary dual modality ∇ in $\mathcal{L}$. This led modal logicians to use modal languages to describe or define classes of relational structures.

In this section, we introduce several classes of structures in which many-valued modal formulas can be interpreted. We then extend the classical constructions of structures to our new classes. Preservation of the validity relation through these constructions routes us to very simple examples of classes of structures that cannot be defined by many-valued modal languages.

The general problem of the characterization of classes of structures by mean of modal formulas can be approached by at least two ways : a model theoric one and an algebraic one. In section 3, we introduce the basic tools for a successful algebraic approach (the one we follow in this dissertation) of the problem : complex and canonical entities. They provide a translation in the algebraic language of validity relations in the structures of different types. So, the tools we introduced here are important building blocks for the sequel of the dissertation.

2.1. Structures. The first structures that we consider are the well known $\mathcal{L}$-frames.

Definition 2.17. An *$\mathcal{L}$-frame* (or simply a *frame*) is a structure $\mathfrak{F} = \langle W, \{R_i \mid i \in I\}\rangle$ where W is a non empty set and R_i is an k_i+1-ary relation on W for any i in I. We denote by $\mathcal{F}r^{\mathcal{L}}$ the class of $\mathcal{L}$-frames. Hence, we use $\mathcal{L}$ to denote a modal language but also a first order language (that contains a k_i+1-relational symbol for ani i in I).

A model $\mathcal{M} = \langle W', \{R'_i \mid i \in I\}, \mathrm{Val}\rangle$ *is based on* a frame $\mathfrak{F} = \langle W, \{R_i \mid i \in I\}\rangle$ if $W = W'$ and $R_i = R'_i$ for any $i \in I$. We extend to frames the vocabulary introduced for models (W is called the *universe* of $\mathfrak{F}$, elements of W are called *worlds*, ...)

An $\mathcal{L}$-formula ϕ is *valid* (resp. *Ł$_n$-valid*) in a state $w \in W$ of a frame $\mathfrak{F} = \langle W, \{R_i \mid i \in I\}\rangle$, in notation $\mathfrak{F}, w \models \phi$ (resp. $\mathfrak{F}, w \models_n \phi$), if $\mathcal{M}, w \models \phi$ for any model $\mathcal{M}$ (resp. any Ł$_n$-valued model) based on $\mathfrak{F}$. This formula is valid in $\mathfrak{F}$ (resp. Ł$_n$-valid in $\mathfrak{F}$), in notations $\mathfrak{F} \models \phi$ (resp. $\mathfrak{F} \models_n \phi$), if ϕ is valid (resp. Ł$_n$-valid) in any state of $\mathfrak{F}$.

Two frames $\mathfrak{F}$ and $\mathfrak{F}'$ are *modally equivalent* (resp. Ł$_n$-modally equivalent) if they validate (resp. Ł$_n$-validate) the same $\mathcal{L}$-formulas.

In sections in which we consider exclusively validity of formulas in frames modulo Ł$_n$-valued models, we replace $\models_n$ by $\models$ and "Ł$_n$-valid" by "valid" to improve readability.

History has proved that, roughly speaking, frames are *the structures of the two-valued normal modal logics* (validity is in that case defined with $\{0,1\}$-valued models). Obviously, thanks to the previous definition, frames can also be used to interpret formulas of $\mathsf{Form}_{\mathcal{L}}$. But, since no information about the many-valued nature of $\mathcal{L}$ is contained in the definition of a frame, it should exists other types of structures that are more efficiently describe by $\mathcal{L}$-formulas. We introduce one of these types of structures in the following definition. Recall that we denote by Ł_0 the MV-algebra $[0,1]$.

DEFINITION 2.18. A *p-$\mathcal{L}$-frame* (or simply a *p-frame*) is a structure

$$\langle W, \{r_m \mid m \in \mathbb{N}_0\}, \{R_i \mid i \in I\}\rangle$$

where

(1) the structure $\langle W, \{R_i \mid i \in I\}\rangle$ is an $\mathcal{L}$-frame,
(2) for any m in $\mathbb{N}_0$, the set r_m is a subset of W and $r_0 = W$,
(3) for any m and k in $\mathbb{N}_0$, we have $r_m \cap r_k = r_{\gcd(m,k)}$.
(4) for any i in I, any m in $\mathbb{N}_0$ and u in r_m, the set $R_i u$ is a subset of $r_m^{k_i}$.

We denote by $\mathcal{PF}r^{\mathcal{L}}$ the class of p-$\mathcal{L}$-frames. If $\mathfrak{F}$ is a p-$\mathcal{L}$-frame, we denote by $\mathfrak{F}_{\#}$ the underlying $\mathcal{L}$-frame of $\mathfrak{F}$.

A model $\mathcal{M} = \langle W', \{R'_i \mid i \in I\}, \mathrm{Val}\rangle$ is *based on a p-$\mathcal{L}$-frame* $\mathfrak{F} = \langle W, \{r_m \mid m \in \mathbb{N}_0\}, \{R_i \mid i \in I\}\rangle$ if $W = W'$, if the relation R_i is equal to R'_i for any i in I and if $\mathrm{Val}(w,p)$ belongs to Ł_m for any w in r_m and any propositional variable p in Prop.

Validity of formulas in p-$\mathcal{L}$-frames is defined similarly as in the case of frames (for example, an $\mathcal{L}$-fromula is valid at w in the p-$\mathcal{L}$-frame $\mathfrak{F}$ if $\mathcal{M}, w \models \phi$ for any model $\mathcal{M}$ based on $\mathfrak{F}$).

Section 4 of this chapter is devoted to a generalization of the well-known duality between complete boolean algebras with complete operators and complete homomorphisms as arrows on the one side and frames on the other side. The structures involved in this generalization are the p-$\mathcal{L}$-frames.

The last classes of structures that we have to introduce are subclasses of the class of p-$\mathcal{L}$-frames. These structures appear throughout the dissertation because in a way, they are more suitable than frames for an algebraic approach of finitely-valued modal logics (see 4.31 for example). From now on, the letter n denotes a positive integer.

DEFINITION 2.19. An *Ł_n-valued $\mathcal{L}$-frame* is a p-$\mathcal{L}$-frame $\mathfrak{F} = \langle W, \{r_m \mid m \in \mathbb{N}_0\}, \{R_i \mid i \in I\}\rangle$ such that $r_n = r_0 = W$. We prefer the notation

$$\langle W, \{r_m \mid m \in \mathrm{div}(n)\}, \{R_i \mid i \in I\}\rangle$$

to denote this frame. We denote by $\mathcal{F}r_n^{\mathcal{L}}$ the class of Ł_n-valued $\mathcal{L}$-frames.

Two Ł_n-valued $\mathcal{L}$-frames are *modally equivalent* if they validate the same $\mathcal{L}$-formulas.

Hence, the class of Ł_n-valued $\mathcal{L}$-frames can equivalently be defined as the class of structures

$$\langle W, \{r_m \mid m \in \mathrm{div}(n)\}, \{R_i \mid i \in I\}\rangle$$

such that

(1) the structure $\langle W, \{R_i \mid i \in I\}\rangle$ is an $\mathcal{L}$-frame,
(2) for any m in $\mathrm{div}(n)$, the set r_m is a subset of W and $r_n = W$,
(3) for any m and k in $\mathrm{div}(n)$, the intersection $r_m \cap r_k$ is equal to $r_{\gcd(m,k)}$,
(4) for any i in I and any m in $\mathrm{div}(n)$, the set $R_i u$ is a subset of $r_m^{k_i}$.

The $Ł_n$-valued $\mathcal{L}$ frames will be extensively used in connection with the $Ł_n$-valued modal logics.

In the sequel, when we write *a class of $\mathcal{L}$-structures* (or simply *a class of structures*), we mean a class of $\mathcal{L}$-frames, or a class of p-$\mathcal{L}$-frames or a class of $Ł_n$-valued $\mathcal{L}$-frames.

2.2. Constructions of structures. In this section, we introduce some ways to construct new structures from existing ones. Each of these constructions is associated with a result about preservation of validity of $\mathcal{L}$-formulas. Thus, they provide a way to produce simple examples of classes of structures that cannot be defined by $\mathcal{L}$-formulas (any class of structures that is not closed for one of these constructions cannot be defined by $\mathcal{L}$-formulas). In section 3.3, we prove that these constructions have natural algebraic translations. This correspondence is deeply used in section 6 in order to obtain the many-valued counterpart of the famous GOLDBLATT - THOMASON theorem about modally definable classes.

We first recall and adapt to the new structures the well known definition of a *bounded morphism*. If R is a $k+1$-ary relation on W, if u belongs to W and if $\psi : W \to W'$ is a map then we denote by $\psi(Ru)$ the set $\{(\psi(v_1), \ldots, \psi(v_k)) \mid (v_1, \ldots, v_k) \in Ru\}$.

DEFINITION 2.20. A map $\psi : \mathfrak{F} \to \mathfrak{F}'$ between two $\mathcal{L}$-frames $\mathfrak{F} = \langle W, \{R_i \mid i \in I\}\rangle$ and $\mathfrak{F}' = \langle W', \{R'_i \mid i \in I\}\rangle$ is a *bounded morphism* if $\psi(R_i(u)) = R'_i(\psi(u))$ for any state u of W.

A map $\psi : \mathfrak{F} \to \mathfrak{F}'$ between two p-$\mathcal{L}$-frames $\mathfrak{F} = \langle W, \{r_m \mid m \in \mathbb{N}_0\}, \{R_i \mid i \in I\}\rangle$ and $\mathfrak{F}' = \langle W', \{r'_m \mid m \in \mathbb{N}_0\}, \{R'_i \mid i \in I\}\rangle$ is a *p-bounded morphism* if ψ is a bounded morphism between the underlying frames of $\mathfrak{F}$ and $\mathfrak{F}'$ and if $\psi(r_m) \subseteq r'_m$ for any m in $\mathbb{N}_0$.

A p-bounded morphism between two $Ł_n$-valued $\mathcal{L}$-frames is called an *$Ł_n$-bounded morphism*.

Sometimes, when we state results about the different types of structures in a same sentence, we use the word *morphism* as a general term to denote these various types of bounded morphisms.

These definition of bounded morphisms are the natural definitions of morphisms between the respective classes of structures that we have previously introduced since they preserve validity of formulas in a way that is precised in the following result.

PROPOSITION 2.21. *Assume that $\mathfrak{F}$ and $\mathfrak{F}'$ are two $\mathcal{L}$-frame (resp. two p-$\mathcal{L}$-frames, two $Ł_n$-valued $\mathcal{L}$-frames). If $\psi : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ is an onto bounded morphism (resp. an onto p-bounded morphism, an onto $Ł_n$-bounded morphism) and if $\mathfrak{F} \models \phi$ then $\mathfrak{F}' \models \phi$.*

PROOF. At this point of the dissertation, the proofs can be considered as exercices. The results will nevertheless follow as consequences of the algebraic treatment of the corresponding constructions of algebras to which section 3 of the present chapter is devoted. □

We can use the previous result to construct examples of non-equational properties, *i.e.* to prove that there are properties (first order properties for example) about these structures that cannot be defined by $\mathcal{L}$-formulas.

DEFINITION 2.22. A class K of $\mathcal{L}$-frames is *$[0,1]$-modally definable* if there is a subset Θ of $\mathsf{Form}_{\mathcal{L}}$ such that $K = \{\mathfrak{F} \in \mathcal{F}r^{\mathcal{L}} \mid \mathfrak{F} \models \Theta\}$. The class K is *$Ł_n$-modally definable* if there is a subset Θ of $\mathsf{Form}_{\mathcal{L}}$ such that $K = \{\mathfrak{F} \in \mathcal{F}r^{\mathcal{L}} \mid \mathfrak{F} \models_n \Theta\}$. A class K of $Ł_n$-valued $\mathcal{L}$-frames is *modally definable* if there is a subset Θ of $\mathsf{Form}_{\mathcal{L}}$ such that $K = \{\mathfrak{F} \in \mathcal{F}r^{\mathcal{L}}_n \mid \mathfrak{F} \models \Theta\}$.

EXAMPLE 2.23. Assume that $\mathcal{L}$ contains just one dual unary modality. The class K of $Ł_n$-valued $\mathcal{L}$-frames that satisfy the first order formula $\forall s(s \notin r_m)$ for a strict divisor m of n is not $Ł_n$-modally definable. Indeed, let us denote by $\mathfrak{F}$ the one irreflexive $Ł_n$-valued $\mathcal{L}$-frame whose universe is $\{s\}$ with s belonging only to r_n and by $\mathfrak{F}'$ the one irreflexive $Ł_n$-valued $\mathcal{L}$-frame whose universe is $\{t\}$ with t belonging to $\bigcup\{r_k \mid k \in \mathrm{div}(n),\, m \in \mathrm{div}(k)\}$.

Then, the $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}'$ is the image of $\mathfrak{F}$ by an $Ł_n$-bounded morphism and the formula $\forall s(s \notin r_m)$ is valid in $\mathfrak{F}$ but is not valid in $\mathfrak{F}'$.

DEFINITION 2.24. If $\psi : \mathfrak{F} \hookrightarrow \mathfrak{F}'$ is a one-to-one bounded morphism between two $\mathcal{L}$-frames $\mathfrak{F}$ and $\mathfrak{F}'$ then ψ is an *embedding*.

If $\psi : \mathfrak{F} \hookrightarrow \mathfrak{F}'$ is a one-to-one p-$\mathcal{L}$-bounded morphism (resp. a one-to-one $Ł_n$-bounded morphism) such that $\psi^{-1} : \psi(\mathfrak{F}) \to \mathfrak{F}$ is also a p-bounded morphism (resp. an $Ł_n$-bounded morphism) then ψ is a *p-embedding* (resp. an *$Ł_n$-embedding*).

An *isomorphism* (resp. *p-isomorphism*, *$Ł_n$-isomorphism*) is an onto p-embedding (resp. onto p-embedding, onto $Ł_n$-embedding).

If $\mathfrak{F}$ is a frame (resp. a p-$\mathcal{L}$-frame, an $Ł_n$-valued $\mathcal{L}$-frame) and if $\mathfrak{F}'$ is a substructure of $\mathfrak{F}$ such that the inclusion map $i : \mathfrak{F}' \hookrightarrow \mathfrak{F}$ is an embedding (resp. a p-embedding, an $Ł_n$-embedding) then $\mathfrak{F}'$ is a *generated subframe* (resp. a *generated p-subframe*, a *generated $Ł_n$-valued subframe*) of $\mathfrak{F}$.

Note that in the definition of a p-embedding, the fact that we require that ψ^{-1} is also a p-embedding means that $\psi^{-1}(r_m^{\psi(\mathfrak{F})}) \subset r_m^{\mathfrak{F}}$. We need to add this condition since it is not satisfied for every one-to-one map.

If we use the notation $\hookrightarrow$ instead of $\to$ in the notation of a map between structures, we mean that the map is, according to the context, an embedding, a p-embedding or an $Ł_n$-embedding.

Once again, we can obtain a preserving result.

PROPOSITION 2.25. *If $\psi : \mathfrak{F} \hookrightarrow \mathfrak{F}'$ is an embedding (resp. a p-embedding, an $Ł_n$-embedding) between two $\mathcal{L}$-frames (resp. two p-$\mathcal{L}$-frames, two $Ł_n$-valued $\mathcal{L}$-frames) $\mathfrak{F}$ and $\mathfrak{F}'$ then for any $\mathcal{L}$-formula ϕ, if $\mathfrak{F}' \models \phi$ then $\mathfrak{F} \models \phi$.*

EXAMPLE 2.26. Assume that $\mathcal{L}$ contains just one unary dual modality. We can use the preserving result to prove that, for any divisor m of n, the class of $Ł_n$-valued $\mathcal{L}$-frames that satisfy the first order formula $\exists s(r_m \subseteq Rs)$ is not $Ł_n$-modally definable. Indeed, assume that $\mathfrak{F}'$ is the $Ł_n$-valued $\mathcal{L}$-frame whose universe is $\{u, v, w\}$ with $R = \{(u, v), (u, w), (u, u)\}$ and $u, v, w \in \bigcup\{r_k \mid m \in \mathrm{div}(k)\}$. If $\mathfrak{F}$ denotes the substructure $\{v, w\}$ of $\mathfrak{F}'$ then $\mathfrak{F}$ is a generated $Ł_n$-valued subframe of $\mathfrak{F}'$. The formula $\exists s(r_m \subseteq Rs)$ is true in $\mathfrak{F}'$ but not in $\mathfrak{F}$.

There are two last constructions that we would like to introduce at this point.

DEFINITION 2.27. A family $\{\mathfrak{F}_j \mid j \in J\}$ of structures is *a family of disjoint structures* if the universe of the structures of the family are pairwise disjoint.

If $\{\mathfrak{F}_j \mid j \in J\}$ is a family of disjoint structures, the *disjoint union of the $\mathfrak{F}_j$ with j in J*, in notation $\biguplus\{\mathfrak{F}_j \mid j \in J\}$ is the structure whose universe is the union for j in J of the universe of $\mathfrak{F}_j$ and whose relations are the unions for j in J of the corresponding relations of the $\mathfrak{F}_j$.

A structure $\mathfrak{F}$ is the *bounded union* of the family $\{\mathfrak{F}_j \mid j \in J\}$ of structures if the universe of $\mathfrak{F}$ is equal to the union for j in J of the universe of the structures $\mathfrak{F}_j$ and if $\mathfrak{F}_j$ is a generated substructure of $\mathfrak{F}$ for any j in J.

If a family $\{\mathfrak{F}_j \mid j \in J\}$ of structures is not disjoint, we can replace each structure $\mathfrak{F}_j$ by an isomorphic copy $\mathfrak{F}_j \times \{j\}$ of $\mathfrak{F}_j$ constructed on $W_j \times \{j\}$ (where W_j is the universe of $\mathfrak{F}_j$) in the obvious way. The family $\{\mathfrak{F}_j \times \{j\} \mid j \in J\}$ is then a family of disjoint structures.

Let us note the almost obvious following results.

PROPOSITION 2.28. *If $\{\mathfrak{F}_j \mid j \in J\}$ is a familly of disjoint $\mathcal{L}$-frames (resp. disjoint p-$\mathcal{L}$-frames, disjoint Ł$_n$-valued $\mathcal{L}$-frames), then $\biguplus\{\mathfrak{F}_j \mid j \in J\} \models \phi$ for any $\mathcal{L}$-formula ϕ such that $\mathfrak{F}_j \models \phi$ for any j in J.*

LEMMA 2.29. *Assume that $\{\mathfrak{F}_j \mid j \in J\}$ is a family of structures.*

1. *The disjoint union and the bounded union of $\{\mathfrak{F}_j \times \{j\} \mid j \in J\}$ coincide.*
2. *If $\mathfrak{F}$ is the bounded union of $\{\mathfrak{F}_j \mid j \in J\}$, then it is an homomorphic image of the disjoint union of $\{\mathfrak{F}_j \times \{j\} \mid j \in J\}$.*

3. Integration of the algebraic ingredient

As suggested by its title, this dissertation is focused on an algebraic view of problems related to validity of formulas in structures and on the links between logics, algebras and structures. We have reached the point where algebras come into action.

3.1. MV-algebras with $\mathcal{L}$-operators. We start by introducing the variety of MV-algebras with dual $\mathcal{L}$-operators. This variety is the many-valued analog of the variety of boolean algebras with dual $\mathcal{L}$-operators. The reader may question about the axiomatization that appears in its definition. One may indeed wonder why we consider this particular generalization of the boolean definition of an operator. But one should keep in mind that we later use this variety to provide a complete algebraic semantic for the many-valued modal logics that we introduce in the next chapter (see Theorem 3.9). This completeness result justifies the definition of the variety.

To improve the readability of the dissertation, we sometimes use a vectorial notation to denote k-uples: the k-uple $(x_1, \ldots, x_k)$ may simply be denoted by $\bar{x}$.

DEFINITION 2.30. Assume that $A, A_1, \ldots, A_k$ are MV-algebras and that $f : A_1 \times \cdots \times A_k \to A$ is a map. Then, for any j in $\{1, \ldots, k\}$, we denote by $f^{(j)}$ the map

$$f^{(j)} : A_j \to A : x \mapsto f((0, \ldots, 0, x, 0, \ldots, 0))$$

where the j^{th}-element of the sequence $(0, \ldots, 0, x, 0, \ldots, 0)$ is equal to x.

A map $\nabla : A_1 \times \cdots \times A_k \to A$ is a k-ary *dual MV-operator* if ∇ is a map

1. that satisfies the axiom (**K**) of modal logic on any of its arguments:
$$\nabla(a_1, \ldots, a_{i-1}, a_i \to b_i, a_{i+1}, \ldots, a_k) \leq \nabla(a_1, \ldots, a_k) \to \nabla(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_k);$$
for any i in $\{1, \ldots, k\}$ and any a_1 in A_1, ..., a_k in A_k and b_i in A_i;
2. that is *conormal*, *i.e.* that satisfies $\nabla^{(j)}(1) = 1$ for any j in $\{1, \ldots, k\}$.

(3) that satisfies

$$\nabla(\bar{a} \oplus \bar{a}) = \nabla\bar{a} \oplus \nabla\bar{a}, \quad \nabla(\bar{a} \odot \bar{a}) = \nabla\bar{a} \odot \nabla\bar{a} \quad \text{and} \quad \nabla(\bar{a} \oplus \bar{a}^m) = \nabla\bar{a} \oplus (\nabla\bar{a})^m$$

for any $\bar{a}$ in $A_1 \times \cdots \times A_k$ and m in $\mathbb{N}_0$.

The map $g : A_1 \times \cdots \times A_k \to A$ is a k-ary *MV-operator* if the map $\bar{g} : A_1 \times \cdots \times A_k \to A : (a_1, \ldots, a_k) \mapsto \neg g(\neg a_1, \ldots, \neg a_k)$ is a dual MV-operator.

A *(complete) dual lattice MV-operator* on a complete MV-algebra A is a dual MV-operator which is a dual (complete) lattice operator. A *complete $\mathcal{L}$-homomorphism* between two complete MV-algebras with dual $\mathcal{L}$-operators is an $\mathcal{L}$-homomorphism which is a complete lattice homomorphism.

An *MV-algebra with dual $\mathcal{L}$-operators* is an algebra A on the language $\mathcal{L}$ such that $\langle A, \oplus, \odot, \neg, 0, 1\rangle$ is an MV-algebra and such that ∇_i is a k_i-ary dual MV-operator for any i in I. The variety of MV-algebras with dual $\mathcal{L}$-operators is denoted by $\mathcal{MMV}^{\mathcal{L}}$. We also denote by $\mathcal{MMV}_n^{\mathcal{L}}$ the subvariety of $\mathcal{MMV}^{\mathcal{L}}$ that contains the algebras whose MV-reduct is in the variety $\mathbb{HSP}(Ł_n)$.

An *MV-algebra with (complete) dual lattice $\mathcal{L}$-operators* is an MV-algebra with dual $\mathcal{L}$-operators such that ∇_i is a (complete) lattice dual MV-operator for any i in I.

To save words, we usually omit the word "dual" in the expression "MV-algebra with dual $\mathcal{L}$-operators". After all, considering MV-operators or dual MV-operators in the definition of $\mathcal{L}$-algebras is much a matter of taste since the definition of an MV-operator and the definition of a dual MV-operator are interdependent.

Note that we provide in Proposition 3.24 a more simple axiomatization of $\mathcal{MMV}_n^{\mathcal{L}}$ in which the equations $\nabla(\bar{x} \oplus \bar{x}^m) = \nabla\bar{x} \oplus (\nabla(\bar{x}))^m$ (with m in $\mathbb{N}_0$) do not appear.

EXAMPLE 2.31. Here are some simple examples of dual MV-operators.

(1) The identity map is a unary dual MV-operator on any MV-algebra A.
(2) The constant map $\mathbf{1} : A \to A : a \mapsto 1$ is a unary dual MV-operator on any MV-algebra A.
(3) $\Box : [0,1] \times [0,1] \to [0,1] : (x, y) \mapsto (\min\{x, y\}, y)$ is a unary dual MV-operator on $[0,1] \times [0,1]$.
(4) If $\mathcal{C}$ denotes CHANG's MV-algebra and if k is a positive integer, then the map $\Box_k : x \mapsto k.x$ is a dual operator on $\mathcal{C}$.

At this point, the reader may note that, even on an MV$_n$-algebra, being a dual lattice operator is not enough to be a dual MV-operator. For example, the dual unary discriminator on $Ł_2$

$$D : x \mapsto \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x < 1 \end{cases}$$

is a dual lattice operator on $Ł_2$ but is not a dual MV-operator (for example $D(\frac{1}{2} \oplus \frac{1}{2}) = 1$ but $D(\frac{1}{2}) \oplus D(\frac{1}{2}) = 0$).

Algebras of $\mathcal{MMV}^{\mathcal{L}}$ can be used to interpret $\mathcal{L}$-formulas.

DEFINITION 2.32. Assume that A is an $\mathcal{MMV}^{\mathcal{L}}$-algebra. An *algebraic valuation on A* is a map $a_{\bullet} : \mathsf{Prop} \to A$. An algebraic valuation $a_{\bullet}$ on A is naturally extended inductively to formulas in the obvious way.

An *algebraic model* $\langle A, a_{\bullet}\rangle$ is given by an $\mathcal{MMV}^{\mathcal{L}}$-algebra A and an algebraic valuation $a_{\bullet}$ on A. A formula ϕ is *true* in an algebraic model $\langle A, a_{\bullet}\rangle$, in notation $\langle A, a_{\bullet}\rangle \models \phi$, if $a_\phi = 1$.

3.2. Canonical and complex entities. In this subsection, we show how MV-algebras with $\mathcal{L}$-operators can be used to encode the information contained in a structure thanks to the construction of a complex algebra. We also prove that it is possible to associate a canonical KRIPKE model to any algebraic model $\langle A, a_{\bullet}\rangle$ in such a way that any true formula in $\langle A, a_{\bullet}\rangle$ is true in its associated model. This result, which is the content of Lemma 2.40, is fundamental in regard to the algebraic approach of the various relational semantics we have previously introduced in this chapter.

3.2.1. *Complex algebras.* We first note that it is possible to associate to any structure an MV-algebra with $\mathcal{L}$-operators in a very natural way, by mimicking the classical construction.

DEFINITION 2.33. If $\mathfrak{F} = \langle W, \{R_i \mid i \in I\}\rangle$ is an $\mathcal{L}$-frame, the *complex algebra* $\mathfrak{F}^+$ of $\mathfrak{F}$ is the algebra

$$\mathfrak{F}^+ = \langle [0,1]^W, \oplus, \neg, (\nabla_i)_{i\in I}, 1\rangle$$

where the operations $\oplus$, $\neg$ and 1 are defined pointwise and where, for every i in I, the operation ∇_i is defined by

$$\nabla_i(\alpha_1, \ldots, \alpha_{k_i})(u) = \bigwedge\{\alpha_1(v_1) \vee \cdots \vee \alpha_{k_i}(v_{k_i}) \mid \bar{v} \in R_i u\}.$$

The $Ł_n$-*complex algebra* of $\mathfrak{F}$ is the algebra

$$\mathfrak{F}^{+_n} = \langle Ł_n^W, \oplus, \neg, (\nabla_i)_{i\in I}, 0, 1\rangle,$$

where the operations are defined as for the complex algebra of a frame.

If $\mathfrak{F} = \langle W, \{r_m \mid m \in \mathbb{N}_0\}, \{R_i \mid i \in I\}\rangle$ is a p-$\mathcal{L}$-frame, the *p-complex algebra* of $\mathfrak{F}$ is the algebra

$$\mathfrak{F}^p = \langle \prod_{u\in W} Ł_{s_u}, \oplus, \neg, (\nabla_i)_{i\in I}, 0, 1\rangle,$$

where $s_u = \gcd\{m \in \mathbb{N} \mid u \in r_m\}$ (so, $s_u = 0$ if $\{m \in \mathbb{N} \mid u \in r_m\}$ is empty) and where operations are defined as for the complex algebra of a frame.

If $\mathfrak{F} = \langle W, \{r_m \mid m \in \operatorname{div}(n)\}, \{R_i \mid i \in I\}\rangle$ is an $Ł_n$-valued $\mathcal{L}$-frame, the $Ł_n$-*tight complex algebra* $\mathfrak{F}^{\times_n}$ of $\mathfrak{F}$ is its p-complex algebra.

The following lemma helps to justify this new vocabulary.

LEMMA 2.34. *Assume that n is a positive integer.*

1. *The complex algebra of a frame is a complete, completely distributive and atomless MV-algebra with dual complete lattice $\mathcal{L}$-operators.*
2. *The p-complex algebra of a frame is a complete and completely distributive MV-algebra with dual complete lattice $\mathcal{L}$-operators.*
3. *The $Ł_n$-complex algebra of a frame and the $Ł_n$-tight complex algebra of an $Ł_n$-valued $\mathcal{L}$-frame are complete, completely distributive and atomic MV_n-algebras with dual complete lattice $\mathcal{L}$-operators.*

PROOF. The proof is just a matter of computation that uses elementary properties of MV-algebras and complete MV-algebras and the continuity of MV-terms when they are interpreted on $[0, 1]$. $\square$

The idea that underlies these constructions is that they translate the concept of validity in the algebraic language. In order to state this result, we use the usual correspondence between $\mathcal{L}$-formulas and $\mathcal{L}$-terms: to any formula ϕ, we associate the $\mathcal{L}$-term ϕ^t whose variables are in $X = \{x_p \mid p \in \mathsf{Prop}\}$ and which is defined inductively by the following rules (we temporally denote by f^t the algebraic operation symbol associated to f for any connective f of $\mathcal{L}$):

- if $p \in \mathsf{Prop}$, we set $p^t = x_p$,
- if $\phi_1, \ldots, \phi_k$ are formulas and f is a k-ary connective of $\mathcal{L}$, the term $(f(\phi_1, \ldots, \phi_k))^t$ is $f^t(\phi_1^t, \ldots, \phi_k^t)$.

Since this definition is clear and natural, if ϕ is an $\mathcal{L}$-formula and if Φ is a set of $\mathcal{L}$-formulas, we simply denote by ϕ the term ϕ^t and by Φ the set $\Phi^t = \{\phi^t \mid \phi \in \Phi\}$ if this convention does not jeopardize the understanding of the results.

LEMMA 2.35. *Assume that ϕ is an $\mathcal{L}$-formula and n is a positive integer.*

(1) *If $\mathfrak{F}$ is a frame, then $\mathfrak{F} \models \phi$ if and only if $\mathfrak{F}^+ \models \phi = 1$.*
(2) *If $\mathfrak{F}$ is a frame, then $\mathfrak{F} \models_n \phi$ if and only if $\mathfrak{F}^{+_n} \models \phi = 1$.*
(3) *If $\mathfrak{F}$ is a p-frame, then $\mathfrak{F} \models \phi$ if and only if $\mathfrak{F}^p \models \phi = 1$.*
(4) *If $\mathfrak{F}$ is an Ł$_n$-frame, then $\mathfrak{F} \models \phi$ if and only if $\mathfrak{F}^{\times_n} \models \phi = 1$.*

PROOF. The proofs of the four results are similar. Let us sketch the proof of the first result. First note that for any MV-algebra with $\mathcal{L}$-operators A, the equation $\phi = 1$ is satsified in A if and only if ϕ is true in any algebraic KRIPKE model $\langle A, a_\cdot \rangle$ based on A.

Let $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val} \rangle$ be a model based on $\mathfrak{F}$ and $a_\cdot$ the algebraic valuation defined on $\mathfrak{F}^+$ by

$$a_p = \mathrm{Val}(\cdot, p)$$

for any p in Prop. It is easy to prove by induction on the number of connectives in ϕ that

$$a_\phi = \mathrm{Val}(\cdot, \phi)$$

for any ϕ in Form. Thus, if $\mathfrak{F}^+ \models \phi = 1$, we obtain that $\langle \mathfrak{F}^+, a_\cdot \rangle \models \phi$ so that $\mathcal{M} \models \phi$.

Conversely, let $\langle \mathfrak{F}^+, a_\cdot \rangle$ be an algebraic model based on $\mathfrak{F}^+$ and $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val} \rangle$ be the model defined by

$$\mathrm{Val}(w, p) = a_p(w)$$

for any propositional variable p and any w in $\mathfrak{F}$. It is easy to prove by induction on the number of connectives in ϕ that

$$\mathrm{Val}(w, \phi) = a_\phi(w)$$

for any ϕ in Form and any w in $\mathfrak{F}$. Hence, if ϕ is valid in $\mathfrak{F}$, it is valid in any algebraic model based on $\mathfrak{F}^+$ and $\mathfrak{F}^+ \models \phi = 1$. □

The following result is clear and does not require a proof.

PROPOSITION 2.36. *Assume that n is a strictly positive integer.*

(1) *If $\mathfrak{F}$ is a frame, then $\mathfrak{F}^{+_n}$ is a complete subalgebra of $\mathfrak{F}^+$ and $\mathfrak{B}(\mathfrak{F}^+)$ coincides with $\mathfrak{B}(\mathfrak{F}^{+_n})$.*
(2) *If $\mathfrak{F}$ is a p-frame, then $\mathfrak{F}^p$ is a complete subalgebra of $\mathfrak{F}^+$ and $\mathfrak{B}(\mathfrak{F}^p)$ coincides with $\mathfrak{B}(\mathfrak{F}^+)$.*
(3) *if $\mathfrak{F}$ is an Ł$_n$-frame, then $\mathfrak{F}^{\times_n}$ is a complete subalgebra of $\mathfrak{F}^{+_n}$ and $\mathfrak{B}(\mathfrak{F}^{\times_n})$ coincides with $\mathfrak{B}(\mathfrak{F}^{+_n})$.*

Actually, albeit obvious, the third statement of Proposition 2.36 is central to our developments about the strong canonicity of SAHLQVIST equations in the variety of MV_n-algebras with $\mathcal{L}$-operators.

3.2.2. *Canonical frames and models.* Complex algebras translate validity of formulas in the algebraic language. We could hope to have a similar construction for the converse problem, *i.e.*, a construction that would associate a first order structure to any MV-algebra with $\mathcal{L}$-operators A in such a way that A would satisfy the equation $\phi = \psi$ if and only if the formula $\phi \leftrightarrow \psi$ is valid in that structure. This project can be categorized as a dream since such a construction is not known for the extensively studied variety of boolean algebras with $\mathcal{L}$-operators, which is a subvariety of $\mathcal{MMV}^{\mathcal{L}}$. But, it is possible to associate a canonical frame A_+ to any boolean algebra with $\mathcal{L}$-operators A in such a way that any $\mathcal{L}$-formula $\phi = \psi$ that is valid in A_+ induces an equation $\phi \leftrightarrow \psi$ which is valid in A. In other words, the frame A_+ only validates formulas that are valid in A.

In this part of the dissertation, we try to generalize this classical construction for the variety $\mathcal{MMV}_{\mathcal{L}}$. Unfortunately, we are able to ensure that the canonical frame A_+ that we associate to A validates formulas that are valid in A only if the MV-reduct of A is an MV_n-algebra for some n in $\mathbb{N}$. The best result we provide for the general case is Lemma 2.40. This result is not about canonical frames but about canonical models. It states that if $\langle A, a_\centerdot\rangle$ is an algebraic valuation and if $\mathcal{M}_{\langle A,a_\centerdot\rangle}$ is the canonical model associated to $\langle A, a_\centerdot\rangle$, then the formula $\phi \leftrightarrow \psi$ is true in $\mathcal{M}_{\langle A,a_\centerdot\rangle}$ if it is true in $\langle A, a_\centerdot\rangle$.

DEFINITION 2.37. If A is an MV-algebra with $\mathcal{L}$-operators, the *canonical frame* A_+ *of* A is the frame

$$A_+ = \langle W_{A_+}, \{R_i^{A_+} \mid i \in I\}\rangle$$

where

(1) the universe W_{A_+} of A_+ is the set $\mathcal{MV}(A, [0,1])$ of the homomorphisms of MV-algebras from A to $[0,1]$;

(2) for any i in I, the relation $R_i^{A_+}$ is defined by

$$(u, v_1, \ldots, v_{k_i}) \in R_i^{A_+} \text{ if } \forall\, a_1, \ldots, a_{k_i} \in A\ (u(\nabla_i(a_1, \ldots, a_{k_i})) = 1 \Rightarrow \bigvee_{1\le k\le k_i} v_k(a_k) = 1)$$

If $a_\centerdot : \mathsf{Prop} \to A$ is an algebraic valuation, the *canonical* KRIPKE-*model associated to the algebraic model* $\langle A, a_\centerdot\rangle$ is the model $\mathcal{M}_{\langle A,a_\centerdot\rangle}$ based on the canonical frame of A and defined by

$$\mathrm{Val}_{\mathcal{M}_{\langle A,a_\centerdot\rangle}}(u, p) = u(a_p)$$

for any propositional variable p and any element u of $\mathcal{MV}(A, [0,1])$.

Note that if A is a boolean algebra with $\mathcal{L}$-operators, these definitions boil down to the standard ones. If A is an algebra of $\mathcal{MMV}_n^{\mathcal{L}}$, the canonical frame of A and the canonical frame of $\mathfrak{B}(A)$ are actually isomorphic.

LEMMA 2.38. *Assume that A is an algebra of $\mathcal{MMV}_n^{\mathcal{L}}$. The map*

$$\psi : A_+ \to \mathfrak{B}(A)_+ : u \mapsto u\restriction_{\mathfrak{B}(A)}$$

is an isomorphism.

PROOF. It is a well known result that ψ is a bijective map (see **[46]** for example). It is also clear that if R is any $k+1$-ary relational symbol associated to a a dual modality of $\mathcal{L}$ and if $(u, v_1, \ldots, v_k)$ belongs to R^A then $(\psi(u), \psi(v_1), \ldots, \psi(v_k))$ belongs to $R^{\mathfrak{B}(A)+}$. Conversely, assume that $(u\restriction_{\mathfrak{B}(A)}, v_1\restriction_{\mathfrak{B}(A)}, \ldots, v_k\restriction_{\mathfrak{B}(A)})$ belongs to $R^{\mathfrak{B}(A)}$ and that $a_1, \ldots, a_k$ are elements of A such that $u(\nabla(a_1, \ldots, a_k)) = 1$. It follows that $u\restriction_{\mathfrak{B}(A)} (\nabla(\tau_1(a_1), \ldots, \tau_1(a_k))) = 1$, so that there is a i in $\{1, \ldots, k\}$ such that $v_i(\tau_1(a_i)) = 1$, which means that $v_i(a_i) = 1$. $\square$

Note that the following proposition provides a definition of R_i^{A+} which is easier to remember, but heavier to check.

PROPOSITION 2.39. *Assume that A is an algebra of $\mathcal{MMV}^{\mathcal{L}}$ and that $i \in I$. An element $(u, v_1, \ldots, v_k)$ of $(A_+)^{k+1}$ belongs to R_i^{A+} if and only if $u(\nabla(x_1, \ldots, x_{k_i})) \leq \bigvee\{v_i(x_i) \mid i \in \{1, \ldots, k_i\}\}$ for any $x_1, \ldots, x_{k_i}$ in A.*

PROOF. The right to left part of the statement is trivial. Let us the left to right part. Proceed *ad absurdum* and assume that $x_1, \ldots, x_k$ are elements of A and that d is a dyadic number of $[0, 1]$ such that

$$\bigvee\{v_i(x_i) \mid i \in \{1, \ldots, k_i\}\} < d \leq u(\nabla_i(x_1, \ldots, x_{k_i})).$$

Then, it follows that

$$\bigvee\{v_i(\tau_d(x_i)) \mid i \in \{1, \ldots, k_i\}\} \neq 1$$

while

$$u(\nabla(\tau_d(x_1), \ldots, \tau_d(x_{k_i})) = 1$$

which is the desired contradiction. $\square$

We now prove that the canonical valuation associated to an algebraic valuation extends naturally to formulas. This lemma is a central result and will allow us to represent any MV_n-algebra with $\mathcal{L}$-operators as a subalgebra of the Ł_n-tight complex algebra of an Ł_n-valued $\mathcal{L}$-frame (see Corollary 2.43).

LEMMA 2.40 (Truth lemma). *Assume that A is an algebra of $\mathcal{MMV}^{\mathcal{L}}$ and that $a_{\bullet} : \mathsf{Prop} \to A$ is an algebraic valuation. If one of the following two conditions is satisfied,*

(1) *the MV-reduct of A is an MV_n-algebra for a positive integer n,*
(2) *every MV-operator of $\mathcal{L}$ is unary,*

then

$$\mathrm{Val}_{\mathcal{M}_{\langle A, a_{\bullet}\rangle}}(u, \phi) = u(a_\phi)$$

for any formula ϕ and any world u of the canonical model $\mathcal{M}_{\langle A, a_{\bullet}\rangle}$ of $\langle A, a_{\bullet}\rangle$.

PROOF. We proceed by induction on the number of connectives of ϕ. The non trivial case is the case of a formula $\phi = \nabla_i(\psi_1, \ldots, \psi_{k_i})$ for an i in I.

It is sufficient to prove that if ∇ is a k-ary dual MV-operator on A, if R denotes the relation associated to ∇ in the canonical frame of A, if u denotes a world of $\mathfrak{F}_A$ and if $a_1, \ldots, a_k$ belong to A, then

$$u(\nabla(a_1, \ldots, a_k)) = \bigwedge_{\bar{v} \in Ru} \bigvee_{1 \leq i \leq k} \bar{v}_i(a_i).$$

We first prove that

$$u(\nabla(a_1,\ldots,a_k)) \leq \bigvee_{1\leq i\leq k} \bar{v}_i(a_i)$$

for any $\bar{v}$ in Ru. Otherwise, there are a $\bar{v}$ in Ru and an element d in $\mathbb{D}\cap[0,1]$ (recall that $\mathbb{D}$ denotes the set of the dyadic numbers) such that

$$\bigvee_{1\leq i\leq k} \bar{v}_i(a_i) < d \leq u(\nabla(a_1,\ldots,a_k)).$$

Thus, for every i in $\{1,\ldots,k\}$, we obtain that

$$\tau_d(\bar{v}_i(a_i)) = \bar{v}_i(\tau_d(a_i)) < 1 \quad \text{and} \quad \tau_d(u(\nabla(a_1,\ldots,a_k))) = 1.$$

If follows that $u(\nabla(\tau_d(a_1),\ldots,\tau_d(a_k))) = 1$ while $\bigvee_{i\leq k}\bar{v}_i(\tau_d(a_i)) < 1$, a contradiction since $\bar{v}$ belongs to Ru.

For the other inequality, we first assume that condition (1) is satisfied. In fact, in that case, the equality

$$u(\nabla(a_1,\ldots,a_k)) = \bigwedge_{\bar{v}\in Ru}\bigvee_{1\leq i\leq k} \bar{v}_i(a_i)$$

is satisfied if and only if, for any i in $\{1,\ldots,n\}$,

$$\tau_{i/n}(u(\nabla(a_1,\ldots,a_k))) = \tau_{i/n}(\bigwedge_{\bar{v}\in Ru}\bigvee_{1\leq i\leq k} \bar{v}_i(a_i)).$$

The latter equality is in turn equivalent to

$$u(\nabla(\tau_{i/n}(a_1),\ldots,\tau_{i/n}(a_k))) = \bigwedge_{\bar{v}\in Ru}\bigvee_{1\leq i\leq k} \bar{v}_i(\tau_{i/n}(a_i)).$$

Eventually, we can conclude the proof of case (1) with the help of the Truth Lemma in boolean algebras with operators and Lemma 2.38.

Let us now assume that condition (2) is satisfied. Then, the considered operator ∇ is unary, and we prefer to denote it by $\Box$. We prove that if there is a d in $\mathbb{D}\cap[0,1]$ and an a in A such that $u(\Box a) < d$, then we can find a w in Ru such that $w(\tau_d(a)) < 1$. If

$$u(\Box a) < d \leq \bigwedge_{v\in Ru} v(a)$$

then for any v in Ru, we obtain $\tau_d(v(a)) = 1$. This condition is not satisfied for the constructed w, a contradiction.

So, proceed *ab absurdum* and suppose that $v(\tau_d(a)) = 1$ for every v in Ru. Since (u,v) belongs to R if and only if $\Box^{-1}u^{-1}(1)\subseteq v^{-1}(1)$, the maximal filters that contain $\Box^{-1}u^{-1}(1)$ are exactly the $v^{-1}(1)$ with v in Ru. If in addition condition (1) is satisfied, we can conclude that $u(\tau_d(a)) = 1$ since in any MV_n-algebra, any filter can be obtained as the intersection of the maximal filters containing it. Otherwise, it follows that the class of $\tau_d(a)$ is infinitely great in $A/\Box^{-1}u^{-1}(1)$ so that $\tau_d(a)\oplus\tau_d(a)^m$ belongs to $\Box^{-1}u^{-1}(1)$ for any positive integer m. Then,

$$1 = u(\Box(\tau_d(a)\oplus\tau_d(a)^m)) = u(\tau_d(\Box a))\oplus u(\tau_d(\Box a))^m$$

for any positive integer m. We conclude that $u(\tau_d(a))$ is inifinitly great in $u(A)$. Since $u(A)$ has no non trivial infinitely great element, we have proved that $u(\tau_d(a)) = 1$, a contradiction. $\Box$

By canonically adding a relational layer to the canonical frame of an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra A, we define the canonical $Ł_n$-valued $\mathcal{L}$-frame associated to A. At this point of the dissertation, the reader may not realize that these structures are more adapted for an algebraic approach of relational semantics of $Ł_n$-valued modal logics than the canonical frames. This will appear clearly in Chapter 4 in which we use this canonical $Ł_n$-valued $\mathcal{L}$-frame to construct a concrete representation of the canonical extension of A (see Proposition 4.31).

Definition 2.41. If A is a member of $\mathcal{MMV}_n^{\mathcal{L}}$, the *canonical $Ł_n$-valued $\mathcal{L}$-frame $A_{\times_n}$ of* A is the structure

$$A_{\times_n} = \langle W_{A_{\times_n}}, \{r_m^{A_{\times n}} \mid m \in \operatorname{div}(n)\}, \{R_i^{A_{\times n}} \mid i \in I\}\rangle$$

where

(1) this structure $\langle W_{A_{\times_n}}, \{R_i^{A_{\times n}} \mid i \in I\}\rangle$ is the canonical frame of A,
(2) for any divisor m of n, the set $r_m^{A_{\times n}}$ contains the homomorphisms that are valued in $Ł_m$:

$$r_m^{A_{\times n}} = \{u \in \mathcal{MV}(A, [0,1]) \mid u(A) \subseteq Ł_m\}.$$

We first prove that canonical $Ł_n$-valued $\mathcal{L}$-frames deserve their names, *i.e.*, that $R_i(r_m^{A_{\times n}}) \subseteq (r_m^{A_{\times n}})^k$.

Lemma 2.42. *Assume that A belongs to $\mathcal{MMV}_n^{\mathcal{L}}$. The structure $A_{\times_n}$ is an $Ł_n$-valued $\mathcal{L}$-frame. As a consequence, the canonical model associated to an algebraic model $\langle A, a_{\bullet}\rangle$ is based on the canonical $Ł_n$-valued $\mathcal{L}$-frame of A.*

Proof. We first prove the result for a unary dual MV-operator $\Box$ with canonical relation R. Let us assume *ab absurdum* that there is a u in $r_m^{A_{\times n}}$ for which the set $Ru \cap X \setminus r_m^{A_{\times n}}$ is not empty. Now, since the subalgebras of $Ł_n$ are the algebras $Ł_m$ with m in $\operatorname{div}(n)$, we can find an m' in $\operatorname{div}(n)$ such that

$$\frac{1}{m'} = \bigwedge\{v(x) \mid v \in Ru \setminus r_m^{A_{\times n}}, x \in A \text{ and } v(x) \neq 0\}.$$

Obviously, the integer m' is not a divisor of m and we can find a $v \in Ru \setminus r_m^{A_{\times n}}$ and a a in A such that $v(a) = \frac{1}{m'}$.

Let us recall that the universe of $A_{\times_n}$ can be equipped with a boolean topology in such a way that the evaluation map

$$e_A : A \hookrightarrow \prod_{u \in A_{\times_n}} u(A) : a \mapsto (u(a))_{u \in A_{\times_n}}$$

is a boolean representation of A such that the set $r_m^{A_{\times n}}$ is a closed set for this topology (see Proposition 1.12). We can thus construct a clopen set Ω containing v and included in $A_{\times_n} \setminus r_m^{A_{\times n}}$. Then the element

$$b = a|_{\Omega} \cup 1|_{A_{\times_n} \setminus \Omega}$$

belongs to A. It follows that

$$u(\Box b) = \bigwedge_{w \in Ru} w(b) = \bigwedge_{w \in Ru \cap \Omega} w(a) = v(a) = \frac{1}{m'}$$

which is a contradiction since $u \in r_m^{A_{\times n}}$.

Let us now consider the case of a k-ary dual MV-operator ∇ with canonical relation R. Suppose *ad absurdum* that there is an element u of $r_m^{A_{\times n}}$ for such that the set

$$R^{\circlearrowleft}u = \bigcup_{i\leq k}\{v \in A_{\times_n} \mid \exists v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k \mid (u,v_1,\ldots,v_{i-1},v,v_{i+1},\ldots,v_k) \in R\}$$

is not a subset of $r_m^{A_{\times n}}$. Then, define m' by

$$\frac{1}{m'} = \bigwedge\{v(x) \mid v \in R^{\circlearrowleft}u \setminus r_m^{A_{\times n}}, x \in A \text{ and } v(x) \neq 0\}.$$

Obviously, m' is not a divisor of m and we can find a v in $R^{\circlearrowleft}u$ and a a in A such that $\frac{1}{m'} = v(a)$. Then, there is a $i \leq k$ and $v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k$ in $A_{\times_n}$ such that

$$(u,v_1,\ldots,v_{i-1},v,v_{i+1},\ldots,v_p) \in R.$$

Let us now consider the unary dual MV-operator

$$\nabla^{(i)} : A \to A : x \mapsto \nabla(0,\ldots,0,x,0,\ldots,0)$$

where x appears as the i^{th} argument of ∇. We prove that (u,v) belongs to $R_{\nabla^{(i)}}$, an absurdity by the unary case since u belongs to $r_m^{A_{\times n}}$ but v does not. If x belongs to A, we obtain

$$\begin{aligned} u(\nabla^{(i)}(x)) = 1 \quad &\Leftrightarrow \quad u(\nabla(0,\ldots,0,x,0,\ldots 0)) = 1 \\ &\Leftrightarrow \quad \bigwedge_{\bar{w}\in Ru} \bar{w}_i(x) = 1, \end{aligned}$$

thanks to Lemma 2.40. Thus, if $u(\nabla^{(i)}(x)) = 1$, we can conclude that $v(x) = 1$ since the $k+1$-uple

$$(u,v_1,\ldots,v_{i-1},v,v_{i+1},\ldots,v_k)$$

belongs to R. $\square$

This result as an important consequence.

COROLLARY 2.43. *If A is an MV_n-algebra with $\mathcal{L}$-operators, then the n-tight complex algebra $(A_{\times_n})^{\times_n}$ of the canonical Ł$_n$-valued $\mathcal{L}$-frame $A_{\times_n}$ of A is an extension of A.*

Similarly as in the case of boolean algebras with operators, we call that extension *the canonical extension of A*. In the sequel of the dissertation, we will show that it is more than a similarity…

DEFINITION 2.44. If A is an MV$_n$-algebra with $\mathcal{L}$-operators, then the algebra $(A_{\times_n})^{\times_n}$, sometimes denoted by A^σ, is called the *canonical extension of A*.

Unfortunately, as proved by the following example, the algebra $(A_+)^+$ is not in general an extension of A.

EXAMPLE 2.45. CHANG's MV-algebra $\mathcal{C}$ is not a a subalgebra of $(\mathcal{C}_+)^+$. Indeed, the algebra $\mathcal{C}$ has only one maximal filter, while $\mathcal{C}$ is not a subalgebra of $[0,1]$.

3.3. Dual constructions. We show that the constructions introduced in subsection 2.2 have a algebraic translations.

PROPOSITION 2.46. *If $\psi : \mathfrak{F} \to \mathfrak{F}'$ is a bounded morphism between two frames $\mathfrak{F}$ and $\mathfrak{F}'$, then the map $\psi^+ : \mathfrak{F}'^+ \to \mathfrak{F}^+ : \alpha \mapsto \alpha \circ \psi$ is a complete $\mathcal{L}$-homomorphism.*

PROOF. Let us assume that $\psi : \mathfrak{F} \to \mathfrak{F}'$ is a bounded morphism. It is easy to prove that ψ^+ is a complete map. If α and β are two elements of $\mathfrak{F}'^+$ and if u is an element of $\mathfrak{F}$, then

$$\begin{array}{rcl}(\psi^+(\alpha \oplus \beta))(u) & = & (\alpha \oplus \beta)(\psi(u)) \\ & = & \alpha(\psi(u)) \oplus \beta(\psi(u)).\end{array}$$

It follows that $\psi^+(\alpha \oplus \beta) = \psi^+(\alpha) \oplus \psi^+(\beta)$. We proceed in a similar way to prove that $\psi(\neg\alpha) = \neg\psi(\alpha)$ for any α in $\mathfrak{F}'^+$.

Now, assume that ∇ is a k-ary dual modality of $\mathcal{L}$. Denote by R and R' the $k+1$-ary relational translations of ∇ in $\mathfrak{F}$ and $\mathfrak{F}'$ respectively. If $\alpha_1, \dots, \alpha_k$ are elements of $\mathfrak{F}'$ and if u belongs to $\mathfrak{F}$, then, on the one hand,

$$\begin{array}{rcl}(\psi^+(\nabla(\alpha_1, \dots, \alpha_k)))(u) & = & (\nabla(\alpha_1, \dots, \alpha_k))(\psi(u)) \\ & = & \displaystyle\bigwedge_{\bar{w} \in R'\psi(u)} \bigvee_{1 \le j \le k} \alpha_j(\bar{w}_j).\end{array}$$

On the other hand,

$$\begin{array}{rcl}(\nabla(\psi^+(\alpha_1), \dots, \psi^+(\alpha_p)))(u) & = & \displaystyle\bigwedge_{\bar{v} \in Ru} \bigvee_{1 \le j \le p} (\psi^+(\alpha_j))(\bar{v}_j) \\ & = & \displaystyle\bigwedge_{\bar{v} \in Ru} \bigvee_{1 \le j \le k} \alpha_j(\psi(\bar{v}_j)).\end{array}$$

The result is then obtained thanks to the definition of a bounded morphism. □

Of course, similar results can be stated for the two other classes of structures.

PROPOSITION 2.47. *Assume that $\psi : \mathfrak{F} \to \mathfrak{F}^+$ is a map between two p-$\mathcal{L}$-frames (resp. two Ł$_n$-valued $\mathcal{L}$-frames) $\mathfrak{F}$ and $\mathfrak{F}'$. Denote by $\psi^p : \mathfrak{F}'^p \to \mathfrak{F}^p$ (resp. by $\psi^{\times_n} : \mathfrak{F}'^{\times_n} \to \mathfrak{F}^{\times_n}$) the map defined by $\psi(\alpha) = \alpha \circ \psi$. If ψ is a p-bounded morphism (resp. an Ł$_n$-bounded morphism) then ψ^p (resp. $\psi^{\times_n}$) is a complete $\mathcal{L}$-homomorphism.*

PROOF. If $\psi : \mathfrak{F} \to \mathfrak{F}'$ is a p-bounded-morphism between two p-$\mathcal{L}$-frames $\mathfrak{F}$ and $\mathfrak{F}'$ then the map ψ^p is an $\mathcal{L}$-homomorphism from $\mathfrak{F}'^+$ to $\mathfrak{F}^+$ according to proposition 2.46. Then, since $\psi(r_m^{\mathfrak{F}}) \subseteq r_m^{\mathfrak{F}'}$ for any m in $\mathbb{N}_0$, the map $\psi\downharpoonleft_{\mathfrak{F}'^p}$ is valued in $\mathfrak{F}^p$ (recall that the p-complex algebra of a frame is a complete subalgebra of its complex algebra). □

We now dualize onto bounded morphisms and embeddings.

PROPOSITION 2.48. *Assume that $\mathfrak{F}$ and $\mathfrak{F}'$ are two $\mathcal{L}$-frames.*

1. *If $\psi : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ is an onto bounded morphism, then $\psi^+ : \mathfrak{F}'^+ \hookrightarrow \mathfrak{F}^+$ is an embedding.*
2. *If $\psi : \mathfrak{F} \hookrightarrow \mathfrak{F}'$ is an embedding, then $\psi^+ : \mathfrak{F}'^+ \twoheadrightarrow \mathfrak{F}^+$ is an onto $\mathcal{L}$-homomorphism.*

A similar result can be stated for p-bounded morphisms (resp. Ł$_n$-valued bounded morphisms) between p-$\mathcal{L}$-frames (resp. Ł-valued $\mathcal{L}$-frames).

PROOF. The proofs are exercices. □

Note that Proposition 2.21 and Proposition 2.25 appear now as consequences of the preceding result and Lemma 2.35.

Here is the dual translation of the disjoint unions of structures.

PROPOSITION 2.49. *If $\{\mathfrak{F}_j \mid j \in J\}$ is a family of disjoint $\mathcal{L}$-frames, then $(\biguplus_{j\in J}\mathfrak{F}_j)^+$ is isomorphic to $\prod_{j\in J}\mathfrak{F}_j^+$. The corresponding result can be stated for disjoint unions of p-$\mathcal{L}$-frames and disjoint unions of $Ł_n$-valued $\mathcal{L}$-frames.*

4. Duality for frames

The preceding developments can be lifted up to a categorical level into a dual equivalence between the category $\mathcal{CMMV}^{\mathcal{L}}$ of complete and completely distributive MV-algebras with complete $\mathcal{L}$-operators on the one hand and the category of p-$\mathcal{L}$-frames on the other hand. Such a duality can also be obtained for the full subcategory $\mathcal{CMMV}_n^{\mathcal{L}}$ of $\mathcal{CMMV}^{\mathcal{L}}$ whose objects are the algebras of $\mathcal{CMMV}^{\mathcal{L}}$ whose MV-reduct is an MV_n-algebra and the category of $Ł_n$-valued $\mathcal{L}$-frames (which is a full subcategory of the category of p-$\mathcal{L}$-frames). These dualities generalize the well known duality between complete and completely distributive boolean algebras with $\mathcal{L}$-operators and the category of $\mathcal{L}$-frames (see [**53**]).

We first fix the notations for the various categories involved.

DEFINITION 2.50. We denote by $\mathcal{CMMV}^{\mathcal{L}}$ the category of complete and completely distributive MV-algebras with complete-$\mathcal{L}$ operators as objects and complete $\mathcal{L}$-homomorphisms as arrows and by $\mathcal{CMMV}_n^{\mathcal{L}}$ the full subcategory of $\mathcal{CMMV}^{\mathcal{L}}$ whose objects are the objects of $\mathcal{CMMV}^{\mathcal{L}}$ whose MV-reduct is an MV_n-algebra.

We denote by $\mathcal{PFr}^{\mathcal{L}}$ the category of p-$\mathcal{L}$-frames as objects and p-bounded-morphisms as arrows and by $\mathcal{Fr}_n^{\mathcal{L}}$ the full subcategory of $\mathcal{PFr}^{\mathcal{L}}$ whose objects are the $Ł_n$-valued $\mathcal{L}$-frames (the arrows of $\mathcal{Fr}_n^{\mathcal{L}}$ are called $Ł_n$-*bounded morphisms* instead of p-bounded morphisms).

In this section we switch our approach of the modalities: we temporarily consider that the language $\mathcal{L}$ is obtained from $\mathcal{L}_{MV}$ by adding connectives Δ_i $(i \in I)$ that are interpreted as MV-operators (instead of connectives ∇_i $(i \in I)$ interpreted as dual MV-operators). This helps to give a nice form to the results.

LEMMA 2.51. *If A is a complete and completely distributive MV-algebra, any complete k-ary MV-operator Δ on A is completely determined by its restriction to $\mathfrak{B}(A)^k$.*

PROOF. In this proof, we identify A with its (completely) isomorphic copy

$$\prod_{p\in\mathsf{Atom}(\mathfrak{B}(A))} i_p((p]),$$

where i_p denotes the unique embedding of $(p]$ in $[0,1]$ (see Lemma 1.20).

Let us define D as the subset of A that contains any element d for which there is a p in $\mathsf{Atom}(\mathfrak{B}(A))$ such that $d(q) = 0$ if $q \neq p$, $d(p)$ belongs to $\mathbb{D} \cap [0,1]$ if $i_p((p]) = [0,1]$ and $d(p) \in Ł_m$ if $i_p((p]) = Ł_m$. Thus, if a belongs to A, then

$$a = \bigvee\{d \mid d \in D \text{ and } d \leq a\}.$$

Since Δ is a complete operator, we obtain that

$$\Delta(a_1,\dots,a_k)=\bigvee\{\Delta(d_1,\dots,d_k)\mid(d_1,\dots,d_k)\in D^k\text{ and }(d_1,\dots,d_k)\leq(a_1,\dots a_k)\}$$

for any $(a_1,\dots,a_k)$ in A^k. Hence, the operator Δ is determined by its value on D^k. Now, to conclude the proof, it suffices to show that if $(d_1,\dots,d_k)$ belongs to D^k and if p is an atom of $\mathfrak{B}(A)$, the value of $(\Delta(d_1,\dots,d_k))(p)$ is determined by $\Delta\restriction_{\mathsf{Atom}(\mathfrak{B}(A))^k}$. Note that

$$(\Delta(d_1,\dots,d_n))(p)=\bigvee\{\sum_{i=1}^{l}c_i^*2^{-i}\mid l\in\mathbb{N}_0,c^*\in\{0,1\}^l\text{ and }g_{c^*}((\Delta(d_1,\dots,d_l))(p))=0\},$$

and that $g_{c^*}((\Delta(d_1,\dots,d_k))(p))=(\Delta(g_{c^*}(d_1),\dots,g_{c^*}(d_k)))(p)$ for any finite sequence c^* of elements of $\{0,1\}$. We can always refine such a sequence c^* in a new sequence c'^* by adding a finite number of 0 to its right end in such a way that $g_{c'^*}(d_i)$ is an atom of $\mathfrak{B}(A)$ for any i in $\{1,\dots,k\}$ and such that the dyadic numbers represented by c^* and c'^* are equal. Thus, $(\Delta(d_1,\dots,d_k))(p)$ is equal to

$$\bigvee\{\sum_{i=1}^{l}c_i^*2^{-i}\mid k\in\mathbb{N}_0,c^*\in\{0,1\}^l,(g_{c^*}(d_1),\dots,g_{c^*}(d_k))\in(\mathsf{Atom}(\mathfrak{B}(A))^k$$
$$\text{and }(\Delta(g_{c^*}(d_1),\dots g_{c^*}(d_k)))(p)=0\},$$

and $(\Delta(d_1,\dots,d_k))(p)$ is effectively determined by $\Delta\restriction_{(\mathsf{Atom}(\mathfrak{B}(A)))^k}$. □

This result indicates us that if we want to construct the analog for a $\mathcal{CMMV}^{\mathcal{L}}$-algebra A of the atomic frame of a complete and completely distributive boolean algebra with operators, it will be sufficient to consider the atomic frame of its algebra of idempotent elements.

Definition 2.52. If A is a complete and completely distributive MV-algebra with complete $\mathcal{L}$-operators, the *atomic frame* A_{a} of A is the atomic frame of $\mathfrak{B}(A)$. Precisely, the frame A_{a} is the frame whose universe is the set $\mathsf{Atom}(\mathfrak{B}(A))$ of atoms of $\mathfrak{B}(A)$ and whose structure is defined by

$$(a,a_1,\dots,a_{k_i})\in R_i\text{ if }a\leq\Delta_i(a_1,\dots,a_{k_i})$$

for any i in I and any $a,b_i,\dots,b_{k_i}$ in $\mathsf{Atom}(\mathfrak{B}(A))$.

Here is a technical lemma which is part of folklore.

Lemma 2.53. *If A and B are two complete and completely distributive boolean algebras and $f:A\to B$ is a complete homomorphism then*

(1) *the map $f_{\mathsf{a}}:\mathsf{Atom}(B)\to A:b\mapsto\bigwedge\{a\in A\mid b\leq f(a)\}$ is valued in $\mathsf{Atom}(A)$,*
(2) *for any b in $\mathsf{Atom}(B)$ there is a unique c in $\mathsf{Atom}(A)$ such that $b\leq f(c)$*
(3) *if $f'_{\mathsf{a}}:\mathsf{Atom}(B)\to\mathsf{Atom}(A)$ denotes the map defined by*

$$f'_{\mathsf{a}}(b)=c\quad\textit{if}\quad b\leq f(c)$$

then $f'_{\mathsf{a}}=f_{\mathsf{a}}$.

Proof. To obtain the first result, it is sufficient to prove that $f_{\mathsf{a}}(b)$ is completely join-prime. Let us assume that $f_{\mathsf{a}}(b)\leq\bigvee X$ for a subset X of A. Then,

$$b\leq f(f_{\mathsf{a}}(b))\leq f(\bigvee X)=\bigvee f(X).$$

Since b is an atom of $\mathfrak{B}(B)$, it follows that there is an x in X such that $b \leq f(x)$. We obtain that $f_{\mathsf{a}}(b) \leq x$ by definition of the map f_{a}.

The second result is trivial: if c and c' are two atoms of A with $b \leq f(c)$ and $b \leq f(c')$, then $b \leq f(c \wedge c') = 0$.

Now, thanks to the second result, the map f_{a} is well defined. Clearly, $f_{\mathsf{a}}(b) \leq f'_{\mathsf{a}}(b)$ for any atom b of B and the other inequality holds because $f_{\mathsf{a}}(b)$ and $f'_{\mathsf{a}}(b)$ are two atoms of A. $\square$

In the following definition, we define the dual of an arrow $f : A \to B$ of $\mathcal{CMMV}^{\mathcal{L}}$ as the dual of its restriction to $\mathfrak{B}(A)$.

Definition 2.54. If $f : A \to B$ is a complete homomorphism between two complete and completely distributive MV-algebras with complete operators A and B, the map $f_{\mathsf{a}} : B_{\mathsf{a}} \to A_{\mathsf{a}}$ is defined by $f_{\mathsf{a}}(b) = \bigwedge\{a \in \mathfrak{B}(A) \mid b \leq f(a)\}$ for any b in $\mathsf{Atom}(\mathfrak{B}(B))$.

To complete the construction of the first layer of the duality (the layer that only involves complete and completely distributive MV-algebras with complete operators and frames), we state the following result.

Lemma 2.55. *If $f : A \to B$ is a complete $\mathcal{L}$-homomorphism between two complete and completely distributive MV-algebras with complete $\mathcal{L}$-operators, then f_{a} is a bounded morphism between B_{a} and A_{a}.*

Proof. The proof follows from the well known corresponding results for the complete and completely distributive boolean algebras with complete operators and complete $\mathcal{L}$-homomorphisms. We nevertheless include a standalone proof.

Let us consider a k-ary MV-operator Δ of $\mathcal{L}$ and $(b, b_1, \ldots, b_k)$ in $R_{\Delta}^{B_{\mathsf{a}}}$. Using the alternative definition of f_{a} proposed in part (3) of Lemma 2.53, we obtain that

$$b \leq f(f_{\mathsf{a}}(b)) \wedge \Delta(b_1, \ldots, b_k).$$

Then, since $b_i \leq f(f_{\mathsf{a}}(b_i))$ for any i in $\{1, \ldots, k\}$, we obtain that

$$\begin{aligned} b &\leq f(f_{\mathsf{a}}(b)) \wedge \Delta(f(f_{\mathsf{a}}(b_1)), \ldots, f(f_{\mathsf{a}}(b_k))) \\ &= f(f_{\mathsf{a}}(b) \wedge \Delta(f_{\mathsf{a}}(b_1), \ldots, f_{\mathsf{a}}(b_k))). \end{aligned}$$

It follows that $f_{\mathsf{a}}(b) \leq \Delta(f_{\mathsf{a}}(b_1), \ldots, f_{\mathsf{a}}(b_k))$ since $f_{\mathsf{a}}(b)$ is an atom and $b \neq 0$.

Let us now assume that b is an atom of $\mathfrak{B}(B)$ and $a_1, \ldots, a_k$ are atoms of $\mathfrak{B}(A)$ such that $(f_{\mathsf{a}}(b), a_1, \ldots, a_k) \in R_{\Delta}^{A_{\mathsf{a}}}$, *i.e.*, such that

$$f_{\mathsf{a}}(b) \leq \Delta(a_1, \ldots, a_k).$$

If we define b_i as $f(a_i)$ for any i in $\{1, \ldots, k\}$, we obtain that

$$b \leq f(f_{\mathsf{a}}(b)) \leq \Delta(b_1, \ldots b_k).$$

Moreover, for any i in $\{1, \ldots, k\}$,

$$\begin{aligned} f_{\mathsf{a}}(b_i) &= \bigwedge\{a \in \mathfrak{B}(A) \mid f(a_i) \leq f(a)\} \\ &\leq a_i, \end{aligned}$$

and so $f_{\mathsf{a}}(b_i) = a_i$ since $f_{\mathsf{a}}(b_i)$ and a_i are two atoms of $\mathfrak{B}(A)$. $\square$

Obviously, the category of $\mathcal{L}$-frames does not contain enough information to be the dual of the category $\mathcal{CMMV}^{\mathcal{L}}$. To get the missing information, we add a layer to the atomic frame of a $\mathcal{CMMV}^{\mathcal{L}}$-algebra A.

DEFINITION 2.56. If A is an algebra of $\mathcal{CMMV}^{\mathcal{L}}$, the *atomic p-$\mathcal{L}$-frame of A*, denoted by A_{p}, is the structure

$$A_{\mathsf{p}} = \langle \mathsf{Atom}(\mathfrak{B}(A)), \{r_m \mid m \in \mathbb{N}_0\}, \{R_i \mid i \in I\}\rangle,$$

where

(1) the structure $\langle \mathsf{Atom}(\mathfrak{B}(A)), \{R_i \mid i \in I\}\rangle$ is the atomic frame of A,
(2) for any m in $\mathbb{N}_0$, the subset r_m contains p if the MV-algebra $(p]$ is embeddable into $Ł_m$, where $Ł_0$ denotes the MV-algebra $[0, 1]$.

Up to now, we can state that if A is a $\mathcal{CMMV}^{\mathcal{L}}$-algebra then the map

$$E_A : A \to (A_{\mathsf{p}})^{\mathsf{P}} : x \mapsto (i_p(x \wedge p))_{p \in \mathsf{Atom}(\mathfrak{B}(A))} \tag{4.1}$$

where i_p denotes the unique embedding from $(p]$ into $[0, 1]$ is an isomorphism of MV-algebras. We can now prove that it also preserves operators of $\mathcal{L}$. Note that this result is well known if A is a boolean algebra with $\mathcal{L}$-operators

Recall that if p is an idempotent element of A, we denote by h_p the onto MV-homomorphism $h_p : A \to (p] : x \mapsto x \wedge p$ (see Lemma 1.18).

DEFINITION 2.57. If A is a complete and completely distributive MV-algebra and if p is an atom of $\mathfrak{B}(A)$, we denote by u_p the map

$$u_p = i_p \circ h_p$$

where i_p denotes the unique MV-embedding of $(p]$ into $[0, 1]$.

We also denote by $J(A)$ the set of the $\vee$-irreducible elements of A. Note that, thanks to isomorphism (4.1), we obtain that $a = \bigvee\{b \mid b \in J(A) \cap (a]\}$ for any a in A.

LEMMA 2.58. *If A belongs to $\mathcal{CMMV}^{\mathcal{L}}$, if Δ is a k-ary modality of $\mathcal{L}$ and if p and $q_1, \ldots, q_k$ are atoms of $\mathfrak{B}(A)$ then $(p, q_1, \ldots, q_k)$ belongs to $R_{\Delta}^{A_{\mathsf{p}}}$ if and only if $(u_p, u_{q_1}, \ldots, u_{q_k})$ belongs to $R_{\Delta}^{A^+}$.*

PROOF. First assume that $(u_p, u_{q_1}, \ldots, u_{q_k})$ belongs to R^{A+}. For any $a_1, \ldots, a_k$ in A if $u_{q_1}(a_1) \wedge \cdots \wedge u_{q_k}(a_k) = 1$ then $u_p(\Delta(a_1, \ldots, a_k)) = 1$ or equivalently if $q_1 \leq a_1, \ldots, q_k \leq a_k$ then $p \leq \Delta(a_1, \ldots, a_k)$. We conclude this part of the proof by considering $a_1 = q_1, \ldots, a_k = q_k$.

Conversely, assume that $(p, q_1, \ldots, q_k) \in R^{A_{\mathsf{p}}}$, *i.e.*, that $p \leq \Delta(q_1, \ldots, q_k)$. Then, if $q_1 \leq a_1, \ldots q_k \leq a_k$, *i.e.*, if $u_{q_1}(a_1) \wedge \cdots \wedge u_{q_k}(a_k) = 1$, we obtain that

$$p \leq \Delta(q_1, \ldots, q_k) \leq \Delta(a_1, \ldots, a_k)$$

and eventually that $u_p(\Delta(a_1, \ldots, a_k)) = 1$ □

PROPOSITION 2.59. *If A is an algebra of $\mathcal{CMMV}^{\mathcal{L}}$ then the map*

$$E_A : A \hookrightarrow (A_{\mathsf{p}})^{\mathsf{P}} : x \mapsto (u_p(x))_{p \in \mathsf{Atom}(\mathfrak{B}(A))}$$

is an $\mathcal{L}$-isomorphism.

PROOF. Let us assume that Δ is a k-ary modality of $\mathcal{L}$. We have to prove that

$$E_A(\Delta^A(a_1,\dots,a_k)) = \Delta^{(A_\mathsf{p})^\mathsf{p}}(E_A(a_1),\dots,E_A(a_k)). \tag{4.2}$$

It is a well known result that this property is satisfied if $(a_1,\dots,a_k)$ belongs to $\mathfrak{B}(A)^k$. Moreover, we have proved in Lemma 2.40 that for any u in A_+,

$$u(\Delta^A(a_1,\dots,a_k)) \geq \bigvee\{v_1(a_1)\wedge\cdots\wedge v_k(a_k) \mid (u,v_1,\dots,v_k)\in R^{A+}\}.$$

Thus, thanks to Lemma 2.58, we obtain that for any p in $\mathsf{Atom}(\mathfrak{B}(A))$,

$$\begin{aligned} E_A(\Delta^A(a_1,\dots,a_n))(p) &\geq \bigvee\{u_{q_1}(a_1)\wedge\cdots\wedge u_{q_k}(a_k) \mid (p,q_1,\dots,q_k)\in R^{A_\mathsf{p}}\} \\ &= (\Delta^{(A_\mathsf{p})^\mathsf{p}}(E_A(a_1),\dots,E_A(a_k)))(p). \end{aligned}$$

We now prove that equation (4.2) holds if $a_1,\dots,a_k$ are $\vee$-irreducible elements of A. Proceed *ad absurdum* and assume that there are $\vee$-irreducible elements $a_1,\dots,a_k$ of A, a p in $\mathsf{Atom}(\mathfrak{B}(A))$ and a d in $\mathbb{D}\cap[0,1]$ such that

$$\bigvee\{u_{q_1}(a_1)\wedge\cdots\wedge u_{q_k}(a_k) \mid (p,q_1,\dots,q_k)\in R^{A_\mathsf{p}}\} < d \leq E_A(\Delta^A(a_1,\dots,a_k)))(p).$$

Since $a_1,\dots,a_k$ are $\vee$-irreducible elements, we can choose (see the remark that follows Lemma 1.20) the term τ_d in such a way that $\tau_d(a_1),\dots,\tau_d(a_k)$ belong to $\mathsf{Atom}(\mathfrak{B}(A))$. It follows that $u_p(\Delta^A(\tau_d(a_1),\dots,\tau_d(a_k))) = 1$ but $u_{q_1}(\tau_d(a_1))\wedge\cdots\wedge u_{q_k}(\tau_d(a_k)) = 0$ if $(p,q_1,\dots,q_k)$ belongs to $R^{(A_\mathsf{p})^\mathsf{p}}$. This is a contradiction since

$$u_p(\Delta^A(\tau_d(a_1),\dots,\tau_d(a_k))) = \bigvee\{u_{q_1}(\tau_d(a_1))\wedge\cdots\wedge u_{q_k}(\tau_d(a_k)) \mid (p,q_1,\dots,q_k)\in R^{(A_\mathsf{p})^\mathsf{p}}\}.$$

It then follows successively that, if p belongs to $\mathsf{Atom}(\mathfrak{B}(A))$ and if $a_1,\dots,a_k$ belong to A

$$\begin{aligned} u_p(\Delta^A(a_1,\dots,a_k)) &= u_p(\bigvee\{\Delta^A(b_1,\dots,b_k) \mid b_1\in(a_1]\cap J(A),\dots,b_k\in(a_k]\cap J(A)\}) \\ &= \bigvee\{u_p(\Delta^A(b_1,\dots,b_k)) \mid b_1\in(a_1]\cap J(A),\dots,b_k\in(a_k]\cap J(A)\} \end{aligned}$$

since u_p is a complete homomorphism. The later element is equal to

$$\bigvee\{\bigvee\{u_{q_1}(b_1)\wedge\cdots\wedge u_{q_k}(b_k) \mid (p,q_1,\dots,q_k)\in R^{(A_\mathsf{p})^\mathsf{p}}\} \mid b_1\in(a_1]\cap J(A),\dots,b_k\in(a_k]\cap J(A)\}$$

and so to

$$\bigvee\{\bigvee\{u_{q_1}(b_1)\wedge\cdots\wedge u_{q_k}(b_k) \mid b_1\in(a_1]\cap J(A),\dots,b_k\in(a_k]\cap J(A)\} \mid (p,q_1,\dots,q_k)\in R^{(A_\mathsf{p})^\mathsf{p}}\}$$

and finally, by complete distributivity of A, to

$$\bigvee\{u_{q_1}(a_1)\wedge\cdots\wedge u_{q_k}(a_k) \mid (p,q_1,\dots,q_k)\in R^{(A_\mathsf{p})^\mathsf{p}}\}$$

which drives us to the desired conclusion. □

The next result justifies the vocabulary introduced in Definition 2.56.

LEMMA 2.60. *If A is a $\mathcal{CMMV}^{\mathcal{L}}$-algebra, then the atomic p-$\mathcal{L}$-frame A_p of A is a p-$\mathcal{L}$-frame.*

Proof. The only non trivial part is to prove that if Δ is an k-ary MV-operator on A with associated relation R on the atomic p-frame of A, if m is a positive integer, if p belongs to r_m and if $(p, q_1, \ldots, q_k)$ belongs to $R^{A_{\mathsf{p}}}$, then $(q_1, \ldots, q_k)$ belongs to $(r_m^{A_{\mathsf{p}}})^k$. We proceed *ad absurdum* and assume that there is a p in $r_m^{A_{\mathsf{p}}}$ and $(q_1, \ldots, q_k)$ in $(R^{A_{\mathsf{p}}})^k$ with q_1 in $A_{\mathsf{p}} \setminus r_m^{A_{\mathsf{p}}}$. Now, let us pick an element x in $(q_1]$ such that $i_{q_1}(x)$ does not belong to Ł_m (where i_{q_1} denotes the embedding defined in the proof of Proposition 2.59) and define the element α_1 of $(A_{\mathsf{p}})^{\mathsf{P}}$ by

$$\alpha_1(r) = \begin{cases} 0 & \text{if } r \neq q_1 \\ x & \text{if } r = q_1, \end{cases}$$

for any r in A_{p}. By definition of $\Delta^{(A_{\mathsf{p}})^{\mathsf{P}}}$, it follows that

$$\begin{aligned} (\Delta^{(A_{\mathsf{p}})^{\mathsf{P}}}(\alpha_1, 1, \ldots, 1))(p) &= \bigvee_{(p, r_1, \ldots, r_k) \in R^{A_{\mathsf{p}}}} \alpha_1(r_1) \\ &= x, \end{aligned}$$

a contradiction since $\alpha(p)$ belongs to Ł_m for any α in $(A_{\mathsf{p}})^{\mathsf{P}}$. $\square$

Recall that we have proved in Lemma in 2.34 that the dual of an object of $\mathcal{PF}r^{\mathcal{L}}$ is an object of $\mathcal{CMMV}^{\mathcal{L}}$ and in Proposition 2.47 that the dual of an arrow of $\mathcal{PF}r^{\mathcal{L}}$ is an arrow of $\mathcal{CMMV}^{\mathcal{L}}$.

We now dualize arrows of $\mathcal{CMMV}^{\mathcal{L}}$ into arrows of $\mathcal{PF}r^{\mathcal{L}}$, *i.e.*, we add a p-frame layer to Lemma 2.55. We use this result as a definition.

Proposition 2.61. *If $f : A \to B$ is a complete $\mathcal{L}$-homomorphism between two $\mathcal{CMMV}^{\mathcal{L}}$-algebras A and B, then the map $f_{\mathsf{a}} : B_{\mathsf{a}} \to A_{\mathsf{a}} : p \mapsto \bigwedge\{a \in \mathfrak{B}(A) \mid p \leq f(a)\}$ is a p-bounded morphism. In order to lift our results at a the level of categories, we denote this map by f_{p}.*

Proof. Thanks to Lemma 2.55, we already know that f_{p} is a bounded morphism between the underlying frames of B_{p} and A_{p}. Let us now assume that p is an atom of $\mathfrak{B}(B)$ that belongs to $r_m^{B_{\mathsf{p}}}$ with $m \in \mathbb{N}_0$. Then, the map

$$g_p : (f_{\mathsf{a}}(p)] \to (p] : x \mapsto f(x) \wedge p$$

is an homomorphism of MV-algebras. Thus, $(p]$ contains a non trivial quotient of $(f_{\mathsf{a}}(p)]$ as a subalgebra (g_p cannot be the constant map 0 otherwise $p = 0$ since $p = f(f_{\mathsf{a}}(p)) \wedge p = g_p(f_{\mathsf{p}}(p))$), which implies that $f_{\mathsf{a}}(p)$ belongs to $r_m^{A_{\mathsf{p}}}$. $\square$

The preceding developments can be turned into a duality result. Actually two duality results.

Proposition 2.62. *The functors $\cdot_{\mathsf{a}}$ and $\cdot^+$ define a dual equivalence between the full subcategory of $\mathcal{CMMV}^{\mathcal{L}}$ whose objects are the atomless objects of $\mathcal{CMMV}^{\mathcal{L}}$ and the category $\mathcal{F}r^{\mathcal{L}}$. The map*

$$E_A : A \to (A_{\mathsf{a}})^+ : a \mapsto (i_p(a \wedge p)))_{p \in \mathsf{Atom}(\mathfrak{B}(A))}$$

(where i_p denotes the unique isomorphism from $(p]$ to $[0,1]$ for any p in $\mathsf{Atom}(\mathfrak{B}(A))$) is an isomorphism for any atomless complete and completely distributive MV-algebra with complete $\mathcal{L}$-opeartors A. The map defined by

$$(\epsilon_{\mathfrak{F}}(p))(q) = \begin{cases} 0 & \text{if } q \neq p \\ 1 & \text{if } q = p \end{cases}$$

is an isomorphism from $\mathfrak{F}$ to $(\mathfrak{F}^+)_+$ for any $\mathcal{L}$-frame $\mathfrak{F}$ and any p and q in $\mathfrak{F}$.

The functors $\cdot_{\mathsf{p}}$ and $\cdot^{\mathsf{p}}$ define a dual equivalence between the category $\mathcal{CMMV}^{\mathcal{L}}$ and the category $\mathcal{PF}r^{\mathcal{L}}$. The map

$$E'_A : A \to (A_{\mathsf{p}})^{\mathsf{p}} : a \mapsto (i'_p(a \wedge p)))_{p \in \mathsf{Atom}(\mathfrak{B}(A))}$$

(where i_p denotes the unique embedding from $(p]$ into $[0,1]$ for any p in $\mathsf{Atom}(\mathfrak{B}(A))$) is an isomorphism for any $\mathcal{CMMV}^{\mathcal{L}}$-algebra A. The map defined by

$$(\epsilon_{\mathfrak{F}}(p))(q) = \begin{cases} 0 & \text{if } q \neq p \\ 1 & \text{if } q = p \end{cases}$$

is an isomorphism from $\mathfrak{F}$ to $(\mathfrak{F}_{\mathsf{p}})^{\mathsf{p}}$ for any $\mathcal{L}$-frame $\mathfrak{F}$ and any p and q in $\mathfrak{F}$.

The restrictions of $\cdot_{\mathsf{p}}$ and $\cdot^{\mathsf{p}}$ to the full subcategory $\mathcal{CMMV}^{\mathcal{L}}_n$ of $\mathcal{CMMV}^{\mathcal{L}}$ and the full subcategory $\mathcal{F}r^{\mathcal{L}}_n$ of $\mathcal{PF}r^{\mathcal{L}}$ respectively define a dual equivalence between these two categories.

PROOF. The easy details are left to the reader. □

5. Canonical extension of structures

There is another construction of structures that we should now introduce. This construction is fundamental for the characterization of classes of structures that are modally definable.

DEFINITION 2.63. If $\mathfrak{F}$ is a frame, the $[0,1]$*-valued canonical extension of* $\mathfrak{F}$, in notation $\mathfrak{F}^{\mathsf{me}_{[0,1]}}$ is the canonical frame $(\mathfrak{F}^+)_+$ of the complex algebra $\mathfrak{F}^+$ of $\mathfrak{F}$. If $\mathcal{M} = \langle \mathfrak{F}, \mathrm{Val}_{\mathcal{M}} \rangle$ is a model based on $\mathfrak{F}$, the *canonical extension of* $\mathcal{M}$ is the model $\mathcal{M}^{\mathsf{me}}$ based on $\mathfrak{F}^{\mathsf{me}_{[0,1]}}$ defined by $\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(w,p) = w(\mathrm{Val}_{\mathcal{M}}(\cdot,p))$ for any propositional variable p and any w in $\mathfrak{F}^{\mathsf{me}_{[0,1]}}$.

If $\mathfrak{F}$ is an $Ł_n$-valued $\mathcal{L}$-frame, the $Ł_n$*-valued canonical extension* of $\mathfrak{F}$, in notation $\mathfrak{F}^{\mathsf{me}_n}$, is the canonical $Ł_n$-valued $\mathcal{L}$-frame $(\mathfrak{F}^{\times_n})_{\times_n}$ associated to its $Ł_n$-tight complex algebra $\mathfrak{F}^{\times_n}$.

If $\mathfrak{F}$ is an $\mathcal{L}$-frame, the $Ł_n$*-valued canonical extension*indexcanonical!$Ł_n$-valued extension of a frame of $\mathfrak{F}$, in notation $\mathfrak{F}^{\mathsf{me}_n}$, is defined as the underlying $\mathcal{L}$-frame of the $Ł_n$-valued canonical extension of the trivial $Ł_n$-valued $\mathcal{L}$-frame based on $\mathfrak{F}$. In other words, the $\mathcal{L}$-frame $\mathfrak{F}^{\mathsf{me}_n}$ is isomorphic to $(\mathfrak{F}^{+_n})_{+_n}$.

If $\mathcal{M}$ is an $Ł_n$-valued $\mathcal{L}$-model, the *underlying* $Ł_n$*-valued* $\mathcal{L}$*-frame* $\mathfrak{F}_n(\mathcal{M})$ of $\mathcal{M}$ is the $Ł_n$-valued $\mathcal{L}$-frame based on form the underlying $\mathcal{L}$-frame of $\mathcal{M}$ by defining the subset r_m of W for any $m \in \mathrm{div}(n)$ by

$$u \in r_m \text{ if } \mathrm{Val}(u, \mathsf{Form}) \subseteq Ł_m.$$

The $Ł_n$*-valued canonical extension* $\mathcal{M}^{\mathsf{me}_n}$ *of* $\mathcal{M}$ is the model based on $\mathfrak{F}_n(\mathcal{M})^{\mathsf{me}_n}$ defined by $\mathrm{Val}_{\mathcal{M}^{\mathsf{me}_n}}(w,p) = w(\mathrm{Val}_{\mathcal{M}}(\cdot,p))$ for any p in Prop and w in $\mathfrak{F}_n(\mathcal{M})^{\mathsf{me}_n}$

Note that if $\mathfrak{F}$ is an $\mathcal{L}$-frame, one can recover the classical (*i.e.*, bi-valued) definition of the canonical extension of $\mathfrak{F}$ by considering $\mathfrak{F}$ as the trivial $Ł_1$-valued $\mathcal{L}$-frame based on $\mathfrak{F}$ and by getting rid of the $Ł_1$-valued layer of the $Ł_1$-valued canonical extension of the latter. We obtain directly the following result.

PROPOSITION 2.64. *If $\mathfrak{F}$ is an $Ł_n$-valued $\mathcal{L}$-frame, the underlying $\mathcal{L}$-frame of $\mathfrak{F}^{\mathsf{me}_n}$ is isomorphic to the $Ł_1$-valued canonical extension of the underlying frame of $\mathfrak{F}$.*

PROPOSITION 2.65. *Assume that $\mathfrak{F}$ is a frame, that $\mathcal{M}$ is a model based on $\mathfrak{F}$ and that ι denotes the map*

$$\iota : \mathfrak{F} \to (\mathfrak{F}^+)_+ : w \mapsto p_w,$$

where p_w denotes the projection map from $\mathfrak{F}^+ = [0,1]^W$ onto its w-th factor.

(1) *The map ι identifies $\mathfrak{F}$ as a subframe of $(\mathfrak{F}^+)_+$,*
(2) *The map ι identifies $\mathcal{M}$ as a submodel of $\mathcal{M}^{\mathsf{me}}$.*

The corresponding result can be stated for the class of Ł_n-valued $\mathcal{L}$-frames and the Ł_n-valued canonical extensions.

PROOF. (1) The map ι is clearly one-to-one. Assume that ∇ is a k-ary dual modality of $\mathcal{L}$ and denote by R (resp. $R^{\mathsf{me}_{[0,1]}}$) its associated relation on $\mathfrak{F}$ (resp. on $(\mathfrak{F}^+)_+$). Let us prove that $\iota(R) = R^{\mathsf{me}_{[0,1]}} \cap \iota(W)^{k+1}$. Note that if $u, v_1, \ldots, v_k$ belong to W, then $(\iota(u), \iota(v_1), \ldots, \iota(v_k))$ belongs to $R^{\mathsf{me}_{[0,1]}}$ if and only if, for every $\alpha_1, \ldots, \alpha_k$ in $[0,1]^W$,

$$p_u(\nabla(\alpha_1, \ldots, \alpha_k)) = 1 \Rightarrow \bigvee_{1 \leq i \leq k} p_{v_i}(\alpha_i) = 1$$

or equivalently, by definition of ∇ on $\mathfrak{F}^+$,

$$\bigwedge_{\bar{w} \in Ru} \bigvee_{1 \leq i \leq k} \alpha_i(\bar{w}_i) = 1 \Rightarrow \bigvee_{1 \leq i \leq k} \alpha_i(v_i) = 1. \tag{5.1}$$

Assume now that $(p_u, p_{v_1}, \ldots, p_{v_k})$ belongs to $\iota(R)$ and that $\bigwedge\{\bigvee\{\alpha_i(\bar{w}_i) \mid 1 \leq i \leq k\} \mid \bar{w} \in Ru\} = 1$. Then, since $\bar{v} \in Ru$, there is a i in $\{1, \ldots, k\}$ such that $p_{v_i}(\alpha_i) = \alpha_i(v_i) = 1$, which proves that $(\iota(u), \iota(v_1) \ldots, \iota(v_k))$ belongs to R^{me}.

To prove the other inclusion, we first consider the case of a unary modal operator $\Box$ (*i.e.*, of a binary accessibility relation R). Assume that $(\iota(u), \iota(v))$ belongs to $R^{\mathsf{me}_{[0,1]}}$. Then, by defining α_1 (in this case $k = 1$) as the characteristic function of Ru in (5.1), we obtain that $\alpha_1(v) = 1$, which means that (u, v) is a member of R. We can now consider the general case of a $k+1$-ary accessibility relation R with $k > 1$. Let us pick an element $(\iota(u), \iota(v_1), \ldots, \iota(v_k))$ in $R^{\mathsf{me}_{[0,1]}}$ and define the binary relations R' and R'' by

$$R'w = \{v \in W \mid (w, v, v_2, \ldots, v_k) \in R\}$$

and

$$R''w = \{v \in W \mid (\iota(w), \iota(v), \iota(v_2), \ldots, \iota(v_k)) \in R^{\mathsf{me}}\}$$

for any w in W.

It suffices to prove that $\iota(R'') \subseteq R'^{\mathsf{me}_{[0,1]}}$. Indeed, in that case, since $(\iota(u), \iota(v_1), \ldots, \iota(v_k))$ belongs to R^{me}, it follows that $(\iota(u), \iota(v_1))$ is a member of $\iota(R'')$ and thus of $R'^{\mathsf{me}_{[0,1]}}$. Now, since R' is a binary relation, we can conclude that (u, v_1) belongs to R' and eventually that $(u, v_1, v_2, \ldots, v_k)$ is in R.

So, let us pick (u, v) in R'' and an α in $[0,1]^W$ such that $\bigwedge\{\alpha(w) \mid w \in R'u\} = 1$. We aim to conclude that $\alpha(v) = 1$. Since (u, v) belongs to R'', we obtain by definition that $(\iota(u), \iota(v), \iota(v_2), \ldots, \iota(v_k))$ is in R^{me}. With the help of (5.1), we conclude that, if $\alpha_1, \ldots, \alpha_k$ are any members of $[0,1]^W$ such that

$$\forall\ \bar{w} \in Ru, \bigvee_{1 \leq i \leq k} \alpha_i(w_i) = 1, \tag{5.2}$$

then $\alpha_1(v) \vee \alpha_2(v_2) \vee \cdots \vee \alpha_k(v_k) = 1$. By defining α_1 as the characteristic function of $R'u$:

$$\alpha_1 : W \to [0,1] : w \mapsto \begin{cases} 1 \text{ if } (u, w, v_2, \ldots, v_k) \in R \\ 0 \text{ if } (u, w, v_2, \ldots, v_k) \notin R, \end{cases}$$

and α_j as the characteristic function of $W \setminus \{v_j\}$:

$$\alpha_j : W \to [0,1] : w \mapsto \begin{cases} 1 \text{ if } w \neq v_j \\ 0 \text{ if } w = v_j, \end{cases}$$

for any $2 \leq j \leq k$, we obtain that $(\alpha_1, \ldots, \alpha_k)$ satisfies condition (5.2). We thus obtain that $\alpha_1(v) \vee \alpha_2(v_2) \vee \cdots \vee \alpha_k(v_k) = 1$. We conclude by definition of the maps α_j (where j is in $\{1, \ldots, k\}$) that $\alpha_1(v) = 1$, so that $(u, v, v_2, \ldots, v_k)$ belongs to R. Thus, we obtain that (u, v) belongs to R' so that $\alpha(v) = 1$ according to our hypothesis on α.

(2) If p is in Prop and w is in W then

$$\begin{aligned} \mathrm{Val}_{\mathcal{M}^{\mathsf{me}_{[0,1]}}}(\iota(w), p) &= \iota(w)(\mathrm{Val}(\cdot, p)) \\ &= \mathrm{Val}(w, p) \end{aligned}$$

which concludes the proof. □

To obtain the following result, we apply Lemma 2.40 which has been obtained for the whole variety $\mathcal{MMV}^{\mathcal{L}}$ only if the language $\mathcal{L}$ does not contain any k-ary dual modality with $k \geq 2$. No such restriction was added for the varieties $\mathcal{MMV}_n^{\mathcal{L}}$ for any n. This asymmetry appears in the following statement.

Lemma 2.66. *Assume that $\mathcal{L}$ is a many-valued language with unary modalities. If $\mathcal{M}$ is a $[0,1]$-valued $\mathcal{L}$-model then*

$$\mathrm{Val}_{\mathcal{M}^{\mathsf{me}_{[0,1]}}}(v', \psi) = v'(\mathrm{Val}_{\mathcal{M}}(\cdot, \psi))$$

for any ψ of $\mathsf{Form}^{\mathcal{L}}$ and any world v' of $\mathcal{M}^{\mathsf{me}_{[0,1]}}$.

The corresponding result can be stated for any language $\mathcal{L}$ of any type and any Ł$_n$-valued $\mathcal{L}$-frame and its Ł$_n$-valued canonical extension $\mathcal{M}^{\mathsf{me}_n}$.

Proof. Let us denote by $\alpha_{\cdot}$ the algebraic valuation on $\mathfrak{F}^+$ defined by

$$\alpha_p = \mathrm{Val}_{\mathcal{M}}(\cdot, p)$$

for any p in Prop. Then, the model $\mathcal{M}^{\mathsf{me}_{[0,1]}}$ appears as the canonical model associated to the algebraic model $\langle \mathfrak{F}^+, \alpha_{\cdot} \rangle$. Thanks to the Lemma 2.40, we obtain that

$$\mathrm{Val}_{\mathcal{M}^{\mathsf{me}_{[0,1]}}}(v', \psi) = v'(\alpha_\psi)$$

for any $\mathcal{L}$-formula ψ and any world v' of $\mathfrak{F}^{\mathsf{me}_{[0,1]}}$. Since it is clear that

$$\mathrm{Val}_{\mathcal{M}}(\cdot, \psi) = \alpha_\psi,$$

we obtain the desired result. □

Proposition 2.67. *Assume that $\mathcal{L}$ is a many-valued modal language with unary modalities. If $\mathcal{M}$ is an $\mathcal{L}$-model then for any world u of $\mathcal{M}$ and any $\mathcal{L}$-formula ϕ,*

$$\mathrm{Val}_{\mathcal{M}}(u, \phi) = \mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(\iota(u), \phi)$$

where ι denotes the map defined in Proposition 2.65.

PROOF. We prove by induction on the number of connectives of ϕ that for any u in $\mathcal{M}$, the truth value $\mathrm{Val}_{\mathcal{M}}(u,\phi)$ is equal to $\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(\iota(u),\phi)$. If ϕ is a propositional variable, the result follows by definition of $\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}$. If $\phi = \neg\psi$ or if $\phi = \psi \oplus \rho$, the proof is easy. Assume now that $\phi = \Box\psi$ and that u is a world of $\mathcal{M}$. It follows successively that

$$\begin{aligned}
\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(\iota(u),\Box\psi) &= \bigwedge\{\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(v',\psi) \mid v' \in R^{\mathsf{me}_{[0,1]}}\iota(u)\}\\
&\leq \bigwedge\{\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(\iota(v),\psi) \mid \iota(v) \in R^{\mathsf{me}_{[0,1]}}\iota(u)\}\\
&= \bigwedge\{\mathrm{Val}_{\mathcal{M}}(v,\psi) \mid \iota(v) \in R^{\mathsf{me}_{[0,1]}}\iota(u)\}\\
&= \mathrm{Val}_{\mathcal{M}}(u,\Box\psi)
\end{aligned}$$

where we have used induction hypothesis in the third step and the fact that ι identifies $\mathfrak{F}$ as a subframe of $\mathfrak{F}^{\mathsf{me}}$ in the last step.

Let us then proceed *ad absurdum* and assume that there is a d in $\mathbb{D}\cap[0,1]$ such that

$$\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(\iota(u),\Box\psi) < d \leq \mathrm{Val}_{\mathcal{M}}(u,\Box\psi). \tag{5.3}$$

Note that a v' in $\mathfrak{F}^{\mathsf{me}_{[0,1]}}$ belongs to $R^{\mathsf{me}}\iota(u)$ if and only if

$$\forall\alpha \in \mathfrak{F}^{+}(\iota(u)(\Box_R\alpha) = 1 \Rightarrow v'(\alpha) = 1)$$

or equivalently if and only if

$$\bigcap\{\iota(v)^{-1}(1) \mid v \in Ru\} \subseteq v'^{-1}(1).$$

From inequations (5.3) we obtain on the one hand that there is a v' in $R^{\mathsf{me}_{[0,1]}}i(u)$ such that $\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(v',\psi) < d$ or equivalently such that $\tau_d(\mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(v',\psi)) \neq 1$. On the other hand, we obtain that $\tau_d(\mathrm{Val}_{\mathcal{M}}(u,\Box\psi)) = 1$, which means that $\tau_d(\mathrm{Val}_{\mathcal{M}}(v,\psi)) = 1$ for any v in Ru.

Since $\tau_d(\mathrm{Val}_{\mathcal{M}}(v,\psi)) = \mathrm{Val}_{\mathcal{M}}(v,\tau_d(\psi)) = i(v)(\mathrm{Val}_{\mathcal{M}}(\cdot,\tau_d(\psi)))$, it follows that

$$\mathrm{Val}_{\mathcal{M}}(\cdot,\tau_d(\psi)) \in \bigcap\{\iota(v)^{-1}(1) \mid v \in Ru\}.$$

Thus, for any v' in $R^{\mathsf{me}_{[0,1]}}\iota(u)$, we can state with the help of Lemma 2.66 that

$$v'(\mathrm{Val}_{\mathcal{M}}(\cdot,\tau_d(\psi))) = \mathrm{Val}_{\mathcal{M}^{\mathsf{me}}}(v',\tau_d(\psi)) = 1.$$

We have thus obtained the desired contradiction. □

Because we have proved Lemma 2.40 for Ł_n-valued $\mathcal{L}$-models where $\mathcal{L}$ contains modal operators of any arity, we can state the following result.

LEMMA 2.68. *If $\mathcal{M}$ is an Ł_n-valued $\mathcal{L}$-model then for any u in $\mathcal{M}$ and any $\mathcal{L}$-formula ϕ,*

$$\mathrm{Val}_{\mathcal{M}}(u,\phi) = \mathrm{Val}_{\mathcal{M}^{\mathsf{me}_n}}(\iota(u),\phi)$$

where ι denotes the map defined in Proposition 2.65.

PROOF. The proof is easily obtained by adapting the proof of the corresponding result for two-valued modal logics with the help of the terms $\tau_{i/n}$. □

COROLLARY 2.69. *Assume that $\mathcal{L}$ is a many-valued modal language with unary operators and that $\mathfrak{F}$ is an $\mathcal{L}$-frame. If ϕ is an $\mathcal{L}$-formula such that $\mathfrak{F}^{\mathsf{me}_{[0,1]}} \models \phi$ then $\mathfrak{F} \models \phi$.*

The corresponding result can be stated for an Ł_n-valued $\mathcal{L}$-frame and its Ł_n-valued canonical extension where $\mathcal{L}$ is any many-valued modal language (without any restriction on the arity of the modalities).

5.1. Canonical $Ł_n$-valued extensions as ultrapowers. A very important result for our algebraic developments is that the $Ł_n$-valued canonical extension of an $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ can be obtained as an ultrapower of $\mathfrak{F}$.

In two-valued modal logic, this result is due to GOLBLATT (see [**25**]). His proof is actually almost sufficient to obtain the desired result for $Ł_n$-valued $\mathcal{L}$-frames. Indeed, if $\mathfrak{F}$ is an $Ł_n$-valued $\mathcal{L}$-frame, his proof provides us with a way to construct the underlying $\mathcal{L}$-frame $(\mathfrak{F}^{\mathsf{me}_n})_\#$ of the $Ł_n$-valued canonical extension of $\mathfrak{F}$ as an ultrapower of $(\mathfrak{F})_\#$. This also explains why this construction cannot be mimic for $[0,1]$-valued canonical extensions.

DEFINITION 2.70. Assume that $\mathcal{L}$ is a first order language and that Y is a set. We denote by $\mathcal{L}_Y$ the language $\mathcal{L} \cup \{P_y \mid y \in Y\}$ where for any y in Y we denote by P_y a unary predicate.

Then, if $\mathfrak{F}$ is a first-order $\mathcal{L}$-model on the universe W (for example if $\mathfrak{F}$ is a frame or an $Ł_n$-valued $\mathcal{L}$-frame) we denote by $\mathfrak{F}_2$ the first order $\mathcal{L}_{2^W}$-model whose $\mathcal{L}$-reduct is equal to $\mathfrak{F}$ and that satisfies

$$\mathfrak{F}_2 \models P_Y(w) \Leftrightarrow w \in Y$$

for any w in W and any subset Y of W.

The first order $\mathcal{L}$-model $\mathfrak{F}$ is *ω-saturated* if for any finite subset Y of the universe of $\mathfrak{F}$ and any set $\Gamma(x)$ of $\mathcal{L}_Y$-formulas with a single free variable x, the set $\Gamma(x)$ is satisfiable in $\mathfrak{F}_2$ if it is finitely satisfiable in $\mathfrak{F}_2$.

THEOREM 2.71. *If $\mathfrak{F}$ is a first-order $\mathcal{L}$-model then there is an elementary extension $\mathfrak{F}_\omega$ of $\mathfrak{F}_2$ which is ω-saturated. Actually, the extension $\mathfrak{F}_\omega$ can be constructed as an ultrapower of $\mathfrak{F}$.*

PROOF. See Theorem 6.1.8 in [**7**]. □

We are now able to prove the desired result. The proof of the following result is a slight adapation to our many-valued settings of the original proof of GOLDBLATT (see [**25**]).

THEOREM 2.72. *If $\mathfrak{F}$ is an $Ł_n$-valued $\mathcal{L}$-frame then the $Ł_n$-valued canonical extension $\mathfrak{F}^{\mathsf{me}_n}$ of $\mathfrak{F}$ can be obtained as an $Ł_n$-valued bounded morphic image of an ultrapower of $\mathfrak{F}$.*

PROOF. In this proof, we identify a subset Y of a set X with its characteristic function on X. Assume that $\mathfrak{F}$ is frame based on W. We prove that $\mathfrak{F}^{\mathsf{me}_n}$ is an $Ł_n$-valued bounded morphic image of the $\mathcal{L}$-reduct of a frame $\mathfrak{F}_\omega$ given by Theorem 2.71. For any element x of $\mathfrak{F}_\omega$ we define $F_x = \{Y \subseteq W \mid \mathfrak{F}_\omega, x \models P_Y(v)\}$. Let us prove that F_x is a prime filter of 2^W for any x in W. Indeed, for any subsets Y and Z of W, the structure $\mathfrak{F}_2$ satisfies the following sentences : $\neg\exists x P_\emptyset(x)$, $\forall x(P_{Y\cap Z}(x) \Leftrightarrow (P_Y(x) \wedge P_Z(x)))$, $\forall x(P_{Y\cup Z}(x) \Rightarrow (P_Y(x) \vee P_Z(x)))$, $\forall x P_W(x)$. Since $\mathfrak{F}_\omega$ is an elementary extension of $\mathfrak{F}_2$, these sentences are also true in $\mathfrak{F}_\omega$ which is sufficient to conclude that F_x is a prime filter of 2^W for any x in $\mathfrak{F}_2$. Hence, there is a unique u_x in $\mathcal{MV}(\mathfrak{F}^{\times_n}, Ł_n)$ such that $u_x^{-1}(1) \cap \mathfrak{B}(\mathfrak{F}^{\times_n}) = F_x$. We are going to prove that the map

$$\psi : \mathfrak{F}_\omega \to \mathfrak{F}^{\mathsf{me}_n} : x \mapsto u_x$$

is an onto $Ł_n$-valued bounded morphism.

Let us first prove that ϕ is an onto map. Assume that u belongs to $\mathfrak{F}^{\mathsf{me}_n}$. Denote by Γ the type

$$\Gamma = \{P_Y \mid u(\chi_Y) = 1\} \cup \{\neg P_Y \mid u(\chi_Y) = 0\}.$$

We prove that Γ is finitely satisfiable in $\mathfrak{F}_2$. Indeed if U and U' are two finite sets of subsets of W such that $u(\chi_Y) = 1$ for any Y in U and $u(\neg\chi_Y) = 1$ for any Y in U', then

$$u(\bigwedge\{\chi_Y \mid Y \in U\}) = u(\bigwedge\{\neg\chi_Y \mid Y \in U'\}) = 1.$$

Hence, there is a w in W such that $\chi_Y(w) = 1$ for any Y in U and such that $\chi_Y(w) = 0$ for any Y in U'. This shows that

$$\{P_Y \mid Y \in U\} \cup \{P_Y \mid Y \in U'\}$$

is satisfiable by w in $\mathfrak{F}_2$. Hence, since $\mathfrak{F}_\omega$ is an elementary extension of $\mathfrak{F}_2$, the type Γ is also finitely satisfiable in $\mathfrak{F}_\omega$. Thanks to the ω-saturation of $\mathfrak{F}_\omega$, we deduce that Γ is satisfiable in $\mathfrak{F}_\omega$. If x is an element of $\mathfrak{F}_\omega$ such that $\mathfrak{F}_\omega, x \models \Gamma$, we obtain that $\psi(x) = u$.

Let us then prove that ψ is a bounded morphism. First assume that $(x, y_1, \ldots, y_k)$ belongs to $R^{\mathfrak{F}_\omega}$ for a $k+1$-ary relation of $\mathcal{L}$. We have to prove that $(\psi(u), \psi(y_1), \ldots, \psi(y_k))$ belongs to $R^{\mathfrak{F}^{\mathsf{me}_n}}$, or equivalently, thanks to Lemma 2.38, that

$$\forall \alpha_1, \ldots, \alpha_k \in 2^W (\psi(x)(\nabla_{R^{\mathfrak{F}}}(\alpha_1, \ldots, \alpha_k) = 1) \Rightarrow \psi(y_1)(\alpha_1) \vee \cdots \vee \psi(y_k)(\alpha_k) = 1).$$

Clearly, by definition of the Ł$_n$-tight complex algebra of $\mathfrak{F}$,

$$\mathfrak{F}_2 \models \forall u \forall v_1 \cdots \forall v_k (P_{\nabla_{R^{\mathfrak{F}}}(\alpha_1,\ldots,\alpha_k)}(u) \wedge (u, v_1, \ldots, v_k) \in R) \Rightarrow (P_{\alpha_1}(v_1) \vee \cdots \vee P_{\alpha_k}(v_k))).$$

Hence, the elementary extension $\mathfrak{F}_\omega$ of $\mathfrak{F}_2$ satisfies that same sentence. Then, for any $\alpha_1, \ldots, \alpha_k$ in 2^W such that $\psi(x)(\nabla_{R^{\mathfrak{F}}}(\alpha_1, \ldots, \alpha_k)) = 1$, we obtain by definition of $\psi(x)$ that

$$\mathfrak{F}_\omega, x \models P_{\nabla_{R^{\mathfrak{F}}}(\alpha_1,\ldots,\alpha_k)}(v),$$

and since $(x, y_1, \ldots, y_k)$ belongs to $R^{\mathfrak{F}_\omega}$, we can conclude that $P_{\alpha_1}(y_1) \vee \cdots \vee P_{\alpha_k}(y_k)$ is true in $\mathfrak{F}_\omega$ which means exactly by definition of ψ that $\psi(y_1)(\alpha_1) \vee \cdots \vee \psi(y_k)(\alpha_k) = 1$.

Let us then assume that $(\psi(z), u_1, \ldots, u_k)$ belongs to $R^{\mathfrak{F}^{\mathsf{me}_n}}$ and prove that there are some $x_1, \ldots, x_k$ in $\mathfrak{F}_\omega$ such that $(z, x_1, \ldots, x_k)$ belongs to $R^{\mathfrak{F}_\omega}$ and that $\psi(x_i) = u_i$ for any i in $\{1, \ldots, k\}$. We first prove that the set of formulas in the variables $v_1, \ldots, v_k$

$$\Gamma = \{R(z, v_1, \ldots, v_k)\} \cup \{P_{Y_1}(v_1) \mid u_1(\chi_{Y_1}) = 1\} \cup \cdots \cup \{P_{Y_k}(v_k) \mid u_k(\chi_{Y_k}) = 1\}$$

is satisfiable in $\mathfrak{F}_\omega$. By ω-saturation of $\mathfrak{F}_\omega$, it is sufficient to prove that Γ is finitely satisfiable in $\mathfrak{F}_\omega$. Since each map u_i's is $\wedge$-preserving, it is equivalent to show that for any subset $Y_1, \ldots Y_k$ of W such that $u_i(\chi_{Y_i}) = 1$ for any i in $\{1, \ldots, k\}$, the set of formulas in $v_1, \ldots, v_k$

$$\Gamma' = \{R(z, v_1, \ldots, v_k), P_{Y_1}(v_1), \ldots, P_{Y_k}(v_k)\}$$

is satisfiable in $\mathfrak{F}_\omega$. Now, since $(\psi(z), u_1, \ldots, u_k)$ belongs to $R^{\mathfrak{F}^{\mathsf{me}_n}}$ and $u_i(\chi_{Y_i}) = 1$ for any i in $\{1, \ldots, k\}$, we obtain by definition of $R^{\mathfrak{F}^{\mathsf{me}_n}}$ that $\psi(z)(\Delta_{R^{\mathfrak{F}}}(\chi_{Y_1}, \ldots, \chi_{Y_k})) = 1$, which means that $\mathfrak{F}_\omega, z \models P_{\Delta_{R^{\mathfrak{F}}}(\chi_{Y_1},\ldots,\chi_{Y_k})}(v)$. Then, since the sentence

$$\forall v (P_{\Delta_{R^{\mathfrak{F}}}(\chi_{Y_1},\ldots,\chi_{Y_k})}(v) \Rightarrow \exists v_1, \ldots, v_k((v, v_1, \ldots, v_k) \in R \wedge P_{Y_1}(v_1) \wedge \cdots \wedge P_{Y_k}(v_k)))$$

is true in $\mathfrak{F}_2$ and so in $\mathfrak{F}_\omega$, we obtain that there are $x_1, \ldots, x_k$ in $\mathfrak{F}_\omega$ such that $(z, x_1, \ldots, x_k) \in R^{\mathfrak{F}_\omega}$ and $\mathfrak{F}_\omega, x_i \models P_{Y_i}(v)$ for any $i \in \{1, \ldots, k\}$ which means that Γ' is satisfiable in $\mathfrak{F}_\omega$.

From the fact that Γ is satisfiable, we obtain that there are some $x_1, \ldots, x_k$ in $\mathfrak{F}_\omega$ such that $(z, x_1, \ldots, x_k)$ belongs to $R^{\mathfrak{F}_\omega}$ and such that $\mathfrak{F}_\omega, x_i \models P_{Y_i}(v)$ for any i in $\{1, \ldots, k\}$ and any Y_i such that $u_i(\chi_{Y_i}) = 1$. We deduce that $\psi(x_i)(\chi_{Y_i}) = 1$ for any i in $\{1, \ldots, k\}$ and any Y_i such that $u_i(\chi_{Y_i}) = 1$. It means that $u_i \lfloor_{\mathfrak{B}(\mathfrak{F}^{\times_n})} = \psi(x_i) \lfloor_{\mathfrak{B}(\mathfrak{F}^{\times_n})}$ and eventually that $\psi(x_i) = u_i$ for any i in $\{1, \ldots, k\}$.

We now prove that $\psi(r_m^{\mathfrak{F}_\omega}) \subseteq r_m^{\mathfrak{F}^{\mathsf{me}_n}}$ for any divisor m of n. So, let us assume that m is a divisor of n, that x belongs to $r_m^{\mathfrak{F}_\omega}$ and that α is an element of the algebra $\mathfrak{F}^{\times_n}$. We have to prove that $\psi(x)(\alpha)$ belongs to Ł_m, or equivalently that $\psi(x)(\tau_{\text{Ł}_m}(\alpha)) = 1$. From the fact that

$$\mathfrak{F}_2 \models \forall v(v \in r_m \Rightarrow P_{\tau_{\text{Ł}_m}(\alpha)}(v))$$

we deduce that the same sentence is true in $\mathfrak{F}_\omega$. Hence, since x belongs to $r_m^{\mathfrak{F}_\omega}$, we conclude that $\mathfrak{F}_\omega, x \models P_{\tau_{\text{Ł}_m}(\alpha)}(v)$, and so, by definition of the map ψ, that $\psi(x)(\tau_{\text{Ł}_m}(\alpha)) = 1$. □

6. Modally definable classes

The previous results can be used to characterize the classes of structures that are modally definable. Our aim is to obtain the equivalent of the Goldblatt - Thomason theorem (recall that this theorem characterize modally definable classes of frames in terms of closure properties). We are going to provide two theorems: a characterization of modally definable classes of Ł_n-valued $\mathcal{L}$-frames and a characterization of Ł_n-modally definable classes of $\mathcal{L}$-frames. The general problem of the characterization of $[0,1]$-modally definable classes of $\mathcal{L}$-frames is unreachable with the tools we developed.

The tools we need to prove these results have been introduced in the previous sections. We can mimic Goldblatt - Thomason's proof. Let us first introduce some notations.

Recall that we denote by n a fixed positive integer. The first group of notations is about constructions associated to *Ł_n-valued $\mathcal{L}$-frames*. The second group is about the corresponding constructions for *$\mathcal{L}$-frames* but with the validity relation $\models_n$ in mind (*i.e.*, the complex algebras that we consider are the Ł_n-complex algebras). To make this asymmetry clear in the notations, we have decided to recall the dependence on n in the first group of notations, but not in the second group.

Definition 2.73. If K is a class of Ł_n-valued $\mathcal{L}$-frames, we denote by

- $\mathbf{Cm}_n(K)$ the class of the Ł_n-tight complex algebras of the structures of K;
- $\mathbf{S}_n(K)$ the class of the generated Ł_n-valued subframes of the structures of K;
- $\mathbf{H}_n(K)$ the class of the Ł_n-valued bounded-morphic images of the structures of K,
- $\mathbf{Ud}(K)$ the class of the disjoint unions of the structures of K,
- $\mathbf{Ex}_n(K)$ the class of the Ł_n-valued canonical extensions of the structures of K;
- $\mathbf{Var}_n(K)$ the variety generated by $\mathbf{Cm}_n(K)$.

If $\mathcal{A}$ is a class of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras, we denote by $\mathbf{Cst}_n(\mathcal{A})$ the class of the canonical Ł_n-valued $\mathcal{L}$-frames of the algebras of $\mathcal{A}$ and by $\mathbf{Str}_n(\mathcal{A})$ the class of the Ł_n-valued $\mathcal{L}$-frames of $\mathcal{A}$, *i.e.*, $\mathbf{Str}_n(\mathcal{A}) = \{\mathfrak{F} \mid \mathfrak{F}^{\times_n} \in \mathcal{A}\}$

If K is a class of $\mathcal{L}$-frames, we denote by

- $\mathbf{Cm}(K)$ the class of the Ł_n-complex algebras of the structures of K;
- $\mathbf{S}(K)$ the class of the generated subframes of the structures of K;
- $\mathbf{H}(K)$ the class of the bounded-morphic images of the structures of K,
- $\mathbf{Ud}(K)$ the class of the disjoint unions of the structures of K,
- $\mathbf{Ex}(K)$ the class of the Ł_n-valued canonical extensions of the structures of K;
- $\mathbf{Var}(K)$ the variety generated by $\mathbf{Cm}(K)$.

If $\mathcal{A}$ is a class of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras, we denote by $\mathbf{Cst}(\mathcal{A})$ the class of the canonical $\mathcal{L}$-frames of the algebras of $\mathcal{A}$ and by $\mathbf{Str}(\mathcal{A})$ the class of $\mathcal{L}$-frames of $\mathcal{A}$, *i.e.*, $\mathbf{Str}(\mathcal{A}) = \{\mathfrak{F} \mid \mathfrak{F}^{\times_n} \in \mathcal{A}\}$

If K is a class of structures, we denote by $\mathbf{Pw}(K)$ the class of the ultrapowers of the structures of K and by $\mathbf{Pu}(K)$ the class of ultraproducts of structures of K.

Theorem 2.75 is the equivalent to the GOLDBLATT - THOMASON theorem (see [**24**], [**25**]). The proof of this theorem is a direct adaptation of the original proof.

LEMMA 2.74. *A class K of Ł$_n$-valued $\mathcal{L}$-frames is modally definable if and only if* $K = \mathbf{Str}_n\mathbf{Var}_n(K)$.

PROOF. First note that any class of Ł$_n$-valued $\mathcal{L}$-frames K is included in $\mathbf{Str}_n\mathbf{Var}_n(K)$.

Then, assume that $K = \{\mathfrak{F} \in \mathcal{F}r_n \mid \mathfrak{F} \models \Phi\}$ for a set of $\mathcal{L}$-formulas Φ. It follows that $\mathbf{Var}_n(K) \models \Phi$, hence that $\mathbf{Str}_n\mathbf{Var}_n(K) \models \Phi$, which means that $\mathbf{Str}_n\mathbf{Var}_n(K) \subseteq K$ as desired.

Conversely, if $K = \mathbf{Str}_n\mathbf{Var}_n(K)$, and if Φ is a set of $\mathcal{L}$-formulas such that Φ^t axiomatizes $\mathbf{Var}_n(K)$, we obtain that

$$K = \mathbf{Str}_n\mathbf{Var}_n(K) = \{\mathfrak{F} \mid \mathfrak{F}^{\times_n} \models \Phi^t\} = \{\mathfrak{F} \mid \mathfrak{F} \models \Phi\} = \mathsf{Mod}(\Phi),$$

and K is definable by Φ. □

THEOREM 2.75. *Assume that K is a class of Ł$_n$-valued $\mathcal{L}$-frames which is closed under taking ultrapowers. Then K is modally definable if and only if the two following conditions are satisfied :*

1. *the class K is closed under taking Ł$_n$-valued generated subframes, disjoint union and Ł$_n$-valued bounded morphic images;*
2. *the class K reflects Ł$_n$-valued canonical extensions: if $\mathbf{Ex}_n(\mathfrak{F})$ belongs to K then $\mathfrak{F}$ belongs to K.*

PROOF. We have already proved the left to right part of the proposition. For the right to left part, it suffices to prove that for any class of Ł$_n$-valued $\mathcal{L}$-frames K,

$$\mathbf{Cst}_n\mathbf{Var}_n(K) \subseteq \mathbf{S}_n\mathbf{H}_n\mathbf{PwUd}(K). \tag{6.1}$$

Indeed, if relation (6.1) is true, it follows that $\mathbf{Cst}_n\mathbf{Var}_n(K) \subseteq K$ since K is closed under taking ultrapowers and satisfies condition (1). Then, for any $\mathfrak{F}$ in $\mathbf{Str}_n\mathbf{Var}_n(K)$, the algebra $\mathfrak{F}^{\times_n}$ belongs to $\mathbf{Var}_n(K)$ and we obtain

$$\mathbf{Ex}_n(\mathfrak{F}) = \mathbf{Cst}_n(\mathfrak{F}^{\times_n}) \in \mathbf{Cst}_n\mathbf{Var}_n(K) \subseteq K.$$

Since K reflects Ł$_n$-valued canonical extensions, we obtain thet $\mathfrak{F}$ belongs to K. It follows that

$$\mathbf{Str}_n\mathbf{Var}_n(K) = K$$

as desired.

So, let us prove that relation (6.1) is true. We obtain that

$$\begin{aligned} \mathbf{Cst}_n\mathbf{Var}_n(K) &= \mathbf{Cst}_n\mathbb{HSP}\mathbf{Cm}_n(K) \\ &= \mathbf{Cst}_n\mathbb{HS}\mathbf{Cm}_n\mathbf{Ud}(K), \end{aligned}$$

where the first equality is obtained by definition of $\mathbf{Var}_n(K)$ and the second one is obtained thanks to Proposition 2.49.

Then, if $\mathfrak{F}$ belongs to $\mathbf{Cst}_n\mathbb{HS}\mathbf{Cm}_n\mathbf{Ud}(K)$, there is a subalgebra A of an algebra of $\mathbf{Cm}_n\mathbf{Ud}(K)$ such that $\mathfrak{F}$ is the canonical structure of an homomorphic image B of A. By the duality theory that we have developed in subsection 3.3, it means that $\mathfrak{F}$ is a generated Ł$_n$-valued subframe of an Ł$_n$-valued bounded morphic image of the Ł$_n$-valued canonical frame associated to an Ł$_n$-tight complex algebra of $\mathbf{Ud}(K)$, *i.e.*, we obtain that

$$\begin{aligned}\mathbf{Cst}_n\mathbb{HS}\mathbf{Cm}_n\mathbf{Ud}(K) &\subseteq \mathbf{S}_n\mathbf{H}_n\mathbf{Str}_n\mathbf{Cm}_n\mathbf{Ud}(K)\\ &= \mathbf{S}_n\mathbf{H}_n\mathbf{Ex}_n\mathbf{Ud}(K).\end{aligned}$$

An application of Theorem 2.72 gives

$$\begin{aligned}\mathbf{S}_n\mathbf{H}_n\mathbf{Ex}_n\mathbf{Ud}(K) &\subseteq \mathbf{S}_n\mathbf{H}_n\mathbf{H}_n\mathbf{PwUd}(K)\\ &= \mathbf{S}_n\mathbf{H}_n\mathbf{PwUd}(K)\end{aligned}$$

which proves relation (6.1). □

One may be tempted to think that there is no need to consider the problem of the characterization of modally definable classes of $\mathcal{L}$-frames separately from the problem of the characterization of modally definable classes of Ł$_n$-valued $\mathcal{L}$-frames. Indeed, roughly speaking, a class of frames can be viewed as a class of trivial Ł$_n$-valued $\mathcal{L}$-frames: an $\mathcal{L}$-frame $\mathfrak{F} = \langle W, \{R_i \mid i \in I\}\rangle$ is Ł$_n$-modally equivalent to the trivial Ł$_n$-valued $\mathcal{L}$-frame based on $\mathfrak{F}$.

So, one may think that to define a class K of $\mathcal{L}$-frames by a set of modal formulas, it suffices to define the class K' of the trivial Ł$_n$-valued $\mathcal{L}$-frames associated to the $\mathcal{L}$-frames of K by a set of modal formulas, *i.e.*, try to find Φ such that $K' = \{\mathfrak{F} \in \mathcal{F}r_n \mid \mathfrak{F} \models \Phi\}$ and you will obtain $K = \{\mathfrak{F} \in \mathcal{F}r \mid \mathfrak{F} \models_n \Phi\}$. This is not true because K may me definable (if K is the class of transitive frames for example) while K' is not definable, because it is not closed under Ł$_n$-bounded morphic images.

Similarly, one may think that K is modally definable by Φ if and only if the class K'' of all the Ł$_n$-valued $\mathcal{L}$-frames whose underlying $\mathcal{L}$-frame belongs to K is definable by Φ. It is for example the case for the class K of transitive frames. The following lemma proves that it

Proposition 2.76. *Assume that K is a class of $\mathcal{L}$-frames and denote by Φ a set of $\mathcal{L}$-formulas and by K'' the class of the Ł$_n$-valued $\mathcal{L}$-frames based on members of K.*

(1) *If $K'' = \{\mathfrak{F} \in \mathcal{F}r_n \mid \mathfrak{F} \models \Phi\}$ then $K = \{\mathfrak{F} \in \mathcal{F}r \mid \mathfrak{F} \models_n \Phi\}$.*
(2) *If $K = \{\mathfrak{F} \in \mathcal{F}r \mid \mathfrak{F} \models_n \Phi\}$ then $K'' \subseteq \{\mathfrak{F} \in \mathcal{F}r_n \mid \mathfrak{F} \models \Phi\}$ but the converse inclusion is not satisfied in general.*

Proof. (1) Assume that $K'' = \{\mathfrak{F} \in \mathcal{F}r_n \mid \mathfrak{F} \models \Phi\}$. If $\mathfrak{F}$ belongs to K then obviously, any model based on $\mathfrak{F}$ can be viewed as a model based on an Ł$_n$-valued $\mathcal{L}$-frame constructed on $\mathfrak{F}$. Hence, we obtain $\mathfrak{F} \models_n \Phi$.

Conversely, if $\mathfrak{F}$ is an $\mathcal{L}$-frame such that $\mathfrak{F} \models_n \Phi$ then Φ is true in the trivial Ł$_n$-valued $\mathcal{L}$-frame $\mathfrak{F}'$ associated to $\mathfrak{F}$. It follows that $\mathfrak{F}'$ belongs to K'', which implies that $\mathfrak{F}$ belongs to K.

(2) The desired inclusion is clear. We provide a counterexample for the other inclusion. Assume that $\mathcal{L}$ contains just one unary dual modality $\Box$ and that $\Phi = \{\Box(p \vee \neg p)\}$. If

we denote by K the class $K = \{\mathfrak{F} \in \mathcal{F}r \mid \mathfrak{F} \models_n \Phi\}$ then K can be described as the class of frames whose accessibility relation is empty. Thus, on the one hand, K'' is the class of the Ł_n-valued $\mathcal{L}$-frames whose accessibility relation is empty. On the other hand, the class $\{\mathfrak{F} \in \mathcal{F}r_n \mid \mathfrak{F} \models \Phi\}$ contains exactly the Ł_n-valued $\mathcal{L}$-frames that satisfy the first order formula $\forall x\ Rx \subseteq r_1$. □

Nevertheless, we can deduce an $\mathcal{L}$-frame version of Theorem 2.75.

PROPOSITION 2.77. *Assume that K is a class of $\mathcal{L}$-frames which is closed under ultrapowers and denote by K'' the class of the Ł_n-valued $\mathcal{L}$-frames whose underlying $\mathcal{L}$-frame belongs to K. The class K is Ł_n-modally definable if and only if K'' is modally definable.*

PROOF. Thanks to the first part of Proposition 2.76, we obtain that if K'' is modally definable then K is Ł_n-modally definable. Now, assume that K is modally definable. Then, the class K is closed under taking generated subframes, bounded-morphic images and disjoint unions and reflects canonical extensions. It follows directly that K'' is closed under taking ultrapowers, Ł_n-valued generated subframes, Ł_n-valued bounded morphic images, disjoint unions and reflects Ł_n-valued canonical extensions. Thanks to Theorem 2.76, we obtain that K'' is modally definable. □

THEOREM 2.78. *Assume that K is a class of $\mathcal{L}$-frames which is closed under taking ultrapowers. Then K is Ł_n-modally definable if and only if the two following conditions are satisfied :*

1. *the class K is closed under taking generated subframes, disjoint unions and bounded morphic images;*
2. *the class K reflects canonical extensions: if $\mathbf{Ex}(\mathfrak{F})$ belongs to K then $\mathfrak{F}$ belongs to K.*

PROOF. We already know that an Ł_n-modally definable class K of $\mathcal{L}$-frames satisfies conditions (1) and (2). Let us prove the converse result.

Thanks to Proposition 2.77, we know that K is Ł_n-modally definable if and only if the class K'' of the Ł_n-valued $\mathcal{L}$-frames based on the $\mathcal{L}$-frames of K is modally definable. Now, since K is closed under ultrapowers and satisfies conditions (1) and (2), it is clear that K'' is closed under ultrapowers and satisfies conditions (1) and (2) of Theorem 2.75. Hence the class K'' is modally definable. □

CHAPTER 3

Many-valued modal systems and completness

In the previous chapters, we have introduced tools to *describe* (properties of) and *define* (classes of) structures. Thus, our approach has been, up to now, a strictly semantical one. No effort has indeed been made to *reason* about frames. This chapter is dedicated to this important side of the modal approach of the various types of structures.

We introduce the many-valued normal modal $\mathcal{L}$-logics, give examples of theorems of these logics, and tackle the problem of completeness of some of these logics with respect to the algebraic semantic and the relational ones.

1. Logics

The modal theories of the $\mathcal{L}$-structures have some part in common. With many-valued normal modal $\mathcal{L}$-logics we intend to provide a way to generate this common part in a syntactic way. The best results (completeness with respect to a finite deductive system) is obtained for $Ł_n$-valued logics. For $[0,1]$-valued logics, we are just able to provide a completeness result for an infinitary deductive system.

Definition 3.1. Assume that $\mathcal{L}$ is a many-valued modal language. A *many-valued modal (normal) $\mathcal{L}$-logic* (or simply an *$\mathcal{L}$-logic* or a *logic*) is a set $\mathbf{L}$ of $\mathcal{L}$-formulas which is closed under the detachment rule (MP), the uniform substitution rule, the necessitation rule (RN) (if $\phi \in \mathbf{L}$ then $\nabla^{(i)}(\phi) \in \mathbf{L}$ for any k-ary dual modality ∇ of $\mathcal{L}$ and any $i \in \{1,\dots,k\}$) and that contains

- an axiomatic base of Łukasiewicz logic:
$$p \to (q \to p), \quad (p \to q) \to ((q \to r) \to (p \to r)),$$
$$((p \to q) \to q) \to ((q \to p) \to p), \quad (\neg p \to \neg q) \to (q \to p)$$
for example;
- the formulas corresponding to the scheme (K) of modal logic:
$$\nabla(p_1,\dots,p_{i-1},p_i \to q_i,p_{i+1},\dots,p_k) \to (\nabla(p_1,\dots,p_k) \to \nabla(p_1,\dots,p_{i-1},q_i,p_{i+1},\dots,p_k))$$
for any k-ary dual modality ∇ of $\mathcal{L}$, and any i in $\{1,\dots,k\}$;
- the formulas
$$\nabla(p_1 \oplus p_1,\dots,p_k \oplus p_k) \leftrightarrow (\nabla(p_1,\dots,p_k) \oplus \nabla(p_1,\dots,p_k))$$
and
$$\nabla(p_1 \odot p_1,\dots,p_k \odot p_k) \leftrightarrow (\nabla(p_1,\dots,p_k) \odot \nabla(p_1,\dots,p_k))$$
for any dual modality ∇ of $\mathcal{L}$;
- the formulas $\nabla(p_1 \oplus p_1^m,\dots,p_k \oplus p_k^m) \leftrightarrow (\nabla(p_1,\dots,p_k) \oplus (\nabla(p_1,\dots,p_k))^m)$ for any k-ary dual modality ∇ of $\mathcal{L}$ and every positive integer m.

- the definitions of the modalities Δ: for any k-ary dual modality ∇ of $\mathcal{L}$, formula

$$\Delta(p_1,\ldots,p_k) \leftrightarrow \neg\nabla(\neg p_1,\ldots,\neg p_k).$$

As usual, we write $\vdash_{\mathbf{L}} \phi$ and say that ϕ is a *theorem of* $\mathbf{L}$ whenever $\phi \in \mathbf{L}$ and denote by $\mathbf{K}^{\mathcal{L}}$, or $\mathbf{K}$ if no confusion on $\mathcal{L}$ is possible, the smallest modal many-valued $\mathcal{L}$-logic.

We often define an $\mathcal{L}$-logic $\mathbf{L}$ by adding a set of axioms Γ to another logic $\mathbf{L}'$. It means that $\mathbf{L}$ is defined as the smallest extension of $\mathbf{L}'$ that contains Γ and that is closed under the rules of substitution, detachment and necessitation. In that case, we denote the logic $\mathbf{L}$ by $\mathbf{L}' + \Gamma$.

For instance, we denote by $\mathbf{K}_n^{\mathcal{L}}$, or simply by $\mathbf{K}_n$, the logic $\mathbf{K} + \Gamma_n$ where Γ_n is made of the formulas $(px^{p-1})^{n+1} \leftrightarrow (n+1)x^p$ for any prime $p < n$ that does not divide n and the formula $(n+1)x \leftrightarrow nx$.

Any extension $\mathbf{L}$ of $\mathbf{K}_n^{\mathcal{L}}$ is called a *modal* $n+1$*-valued* $\mathcal{L}$*-logic* or simply a *modal* $Ł_n$*-valued logic*.

The logic $\mathbf{K}_n^{\mathcal{L}_1}$ (where $\mathcal{L}_1$ denote the basic modal language that contains just one unary dual modality $\Box$) is a generalization of a logic introduced in [**48**].

We can obviously associate to each many-valued modal logic an Hilbert system defined in a natural way.

In order to become acquainted with these modal many-valued logics, it is worth to give some basic examples of theorems. For sake of readability, we provide these theorems for the basic modal language $\mathcal{L}_1$.

Proposition 3.2. *The following formulas are theorems of* $\mathbf{K}^{\mathcal{L}_1}$*:*

(1) $\Box(p \to q) \to (\Diamond p \to \Diamond q)$,
(2) $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$,
(3) $\Box p \odot \Box q \to \Box(p \odot q)$
(4) $\Diamond(p \oplus q) \to (\Diamond p \oplus \Diamond q)$
(5) $\Box(p \wedge q) \to (\Box p \wedge \Box q)$,
(6) $\Diamond p \vee \Diamond q \to \Diamond(p \vee q)$.

Moreover, the logic $\mathbf{K}$ *is closed under the following deduction rules:*

(1) $\dfrac{p \leftrightarrow q}{\Box p \leftrightarrow \Box q}$

(2) $\dfrac{\phi_1 \odot \cdots \odot \phi_k \to \psi}{\Box\phi_1 \odot \cdots \odot \Box\phi_k \to \Box\psi}$

Proof. We provide a proof of (1).

(1.1)	$\mathbf{K}^{\mathcal{L}_1}$	$\vdash$	$(p \to q) \to (\neg q \to \neg p)$ (contraposition)
(1.2)		$\vdash$	$\Box((p \to q) \to (\neg q \to \neg p))$ (necessitation rule (1.1))
(1.3)		$\vdash$	$\Box((p \to q) \to (\neg q \to \neg p)) \to (\Box(p \to q) \to (\Box\neg q \to \Box\neg p))$ ($\mathbf{K}$)
(1.4)		$\vdash$	$\Box(p \to q) \to (\Box\neg q \to \Box\neg p)$ (detachment (1.2), (1.3))
(1.5)		$\vdash$	$(\neg\Diamond q \to \neg\Diamond p) \to (\Diamond p \to \Diamond q)$ (converse contraposition)

(1.6) $\quad \vdash \quad (\Box(p \to q) \to (\neg\Diamond q \to \neg\Diamond p)) \to [((\neg\Diamond q \to \neg\Diamond p) \to (\Diamond p \to \Diamond q))$

(1.7) $\quad \to (\Box(p \to q) \to (\Diamond p \to \Diamond q))]$ (transitivity of $\to$)

(1.8) $\quad \vdash \quad (\Box(p \to q) \to (\Diamond p \to \Diamond q))$ (double detachment (1.4), (1.5), (1.7)).

The other theorems and the deduction rules are left as exercises. $\Box$

We claim that many-valued modal logics are suitable to reason about many-valued models and frames. The first step in the process of proving that claim consists in convincing us that we cannot derive as a theorem of $\mathbf{K}$ a formula that is not valid in the class of $[0,1]$-valued KRIPKE models.

THEOREM 3.3 (Soundness). *Assume that $\mathcal{L}$ is a many-valued modal language.*

(1) *If ϕ is a theorem of $\mathbf{K}^{\mathcal{L}}$ and if $\mathcal{M}$ is a $[0,1]$-valued* KRIPKE *model, then $\mathcal{M} \models \phi$.*
(2) *Similarly, if ϕ is a theorem of $\mathbf{K}_n^{\mathcal{L}}$ and if $\mathcal{M}$ is an $Ł_n$-valued* KRIPKE *model, then $\mathcal{M} \models \phi$.*

PROOF. We have already proved in Proposition 2.7 that the axioms of $\mathbf{K}^{\mathcal{L}}$ and $\mathbf{K}_n^{\mathcal{L}}$ are valid in any $[0,1]$-valued KRIPKE model and any $Ł_n$-valued KRIPKE model respectively. To conclude the proof, we just note that if ϕ and $\phi \to \psi$ are true in a class M of models, then ψ is also true in M, that if ϕ is true in M, then $\Box\phi$ is true in M and eventually that if $\phi(p_1, \ldots p_k)$ is true in M, then $\phi(\psi_1, \ldots, \psi_k)$ is true in M for any formulas $\psi_1, \ldots, \psi_k$. $\Box$

We can define a notion of deduction from a set of assumptions Γ in the usual way.

DEFINITION 3.4. If $\mathbf{L}$ is a many-valued modal logic and $\Gamma \cup \{\phi\}$ is a set of formulas, then *ϕ is deducible from Γ in $\mathbf{L}$*, in notation $\Gamma \vdash_{\mathbf{L}} \phi$, if ϕ belongs to the smallest extension of $\mathbf{L}$ that is closed under substitution and detachment.

Note that any many-valued modal logic $\mathbf{L}$ coincides with the set of formulas that are deducible in $\mathbf{L}$ from the empty set.

For the readers that prefer to use the HILBERT system associated to $\mathbf{L}$, note that a formula ϕ is deducible from Γ in $\mathbf{L}$ if and only if there is a proof of ϕ in the system associated to $\mathbf{L}$ in which the necessitation rule is never applied to a formula that is dependent on Γ (where a formula is *dependent on* Γ if it belongs to Γ or if it is obtained by discharging a formula that depends on Γ).

2. The algebraic semantic

Here we apply the classical LINDENBAUM-TARSKI construction to the many-valued modal logics in order to incorporate the algebraic tool in our approach of the problem of completeness of these logics.

Before reading this definition, recall the conventions about algebraic terms and formulas that we have set up in our remark that follows Lemma 2.34.

DEFINITION 3.5. For any many-valued modal $\mathcal{L}$-logic $\mathbf{L}$, we denote by $\mathcal{MMV}_{\mathbf{L}}^{\mathcal{L}}$ the variety

$$\mathcal{MMV}_{\mathbf{L}}^{\mathcal{L}} = \{A \in \mathcal{MMV}^{\mathcal{L}} \mid \forall \phi \in \mathbf{L},\ A \models \phi = 1\},$$

that we call the *variety of $\mathbf{L}$-algebras*.

Note that $\mathcal{MMV}^{\mathcal{L}} = \mathcal{MMV}^{\mathcal{L}}_{\mathbf{K}}$ and that $\mathcal{MMV}^{\mathcal{L}}_{n} = \mathcal{MMV}^{\mathcal{L}}_{\mathbf{K}_n}$.

LEMMA 3.6. *Assume that A is an $\mathcal{MMV}^{\mathcal{L}}$-algebra and $\phi(p_1, \ldots, p_k)$ is an $\mathcal{L}$-formula.*

(1) *If $a_{\bullet}$ is an algebraic valuation on A then $\langle A, a_{\bullet}\rangle \models \phi$ if and only if the equation $\phi^t(a_{p_1}, \ldots, a_{p_k}) = 1$ is satisfied in A.*
(2) *$A \models \phi$ if and only if $\langle A, a_{\bullet}\rangle \models \phi$ for any algebraic valuation $a_{\bullet}$ on A.*

PROOF. The proof is a routine argument. □

DEFINITION 3.7. If $\mathbf{L}$ is a many-valued modal $\mathcal{L}$-logic and if X is a set, we denote by $\mathcal{F}_{\mathbf{L}}(X)$ the *free* $\mathbf{L}$*-algebra over the set X of generators*, *i.e.*, the algebra $\mathcal{F}_{\mathbf{L}}(X)$ is the quotient of the algebra of the $\mathcal{L}$-terms whose variables are in X by the syntactic equivalence relation $\equiv_{\mathbf{L}}$ which is defined by

$$\phi^t \equiv_{\mathbf{L}} \psi^t \quad \text{if} \quad \mathbf{L} \vdash \phi \leftrightarrow \psi.$$

We simply denote by $\mathcal{F}_{\mathbf{L}}$ the free $\mathbf{L}$-algebra over a enumerable set of generators. For any formula ϕ, we denote by $\phi^{\mathbf{L}}$ the element $\phi^t/\equiv_{\mathbf{L}}$ of $\mathcal{F}_{\mathbf{L}}$. If Γ is a set of formulas, we naturally denote by $\Gamma^{\mathbf{L}}$ the subset $\{\phi^{\mathbf{L}} \mid \phi \in \Gamma\}$ of $\mathcal{F}_{\mathbf{L}}$.

Recall that if Y is a subset of an MV-algebra A, we denote by $\langle Y\rangle$ the implicative filter of A generated by Y.

PROPOSITION 3.8. *Assume that $\mathbf{L}$ is a many-valued modal $\mathcal{L}$-logic and that $\Gamma \cup \{\phi\}$ is a set of $\mathcal{L}$-formulas. Then $\Gamma \vdash_{\mathbf{L}} \phi$ if and only if $\phi^{\mathbf{L}}/\langle\Gamma^{\mathbf{L}}\rangle = 1$ in $\mathcal{F}_{\mathbf{L}}/\langle\Gamma^{\mathbf{L}}\rangle$.*

The preceding Proposition is particularly interesting when $\Gamma = \emptyset$ because it provides us with a completeness result.

THEOREM 3.9. *Assume that $\mathbf{L}$ is a many-valued modal $\mathcal{L}$-logic and that ϕ is an $\mathcal{L}$-formula. The formula ϕ is a theorem of $\mathbf{L}$ if and only if ϕ is valid in any $\mathbf{L}$-algebra.*

DEFINITION 3.10. If $\mathcal{A}$ is a class of $\mathcal{L}$-algebras, we denote by $\mathrm{Log}(\mathcal{A})$ the set

$$\mathrm{Log}(\mathcal{A}) = \{\phi \in \mid \mathcal{A} \models \phi = 1\}.$$

If $\mathbf{L}$ is a many-valued modal $\mathcal{L}$-logic, we denote by $\mathsf{Next}(\mathbf{L})$ the lattice of the *normal extensions* of $\mathbf{L}$, that is, $\mathsf{Next}(\mathbf{L})$ is made of the many-valued modal $\mathcal{L}$-logics that contain $\mathbf{L}$ and is ordered by inclusion.

If $\mathcal{A}$ is a variety of $\mathcal{MMV}^{\mathcal{L}}$-algebras, then we denote by $\mathsf{SubVar}(\mathcal{A})$ the lattice of the subvarieties of $\mathcal{A}$.

We can state the following result which may be considered as a good starting point for the study of the lattice of many-valued modal $\mathcal{L}$-logics. It is nevertheless not our purpose to initiate such a study in this dissertation.

PROPOSITION 3.11. *The maps $\mathrm{Log}(\cdot) : \mathsf{SubVar}(\mathcal{MMV}^{\mathcal{L}}) \to \mathsf{Next}(\mathbf{K}_{\mathcal{L}})$ and $\mathcal{MMV}^{\mathcal{L}}_{\cdot} : \mathsf{Next}(\mathbf{K}_{\mathcal{L}}) \to \mathsf{SubVar}(\mathcal{MMV}^{\mathcal{L}})$ are two dual lattice isomorphisms that are inverse to each other.*

Note that we can still characterize congruences by means of subsets.

DEFINITION 3.12. Assume that $\mathcal{L}$ is a many-valued modal language. We define a family $C^{\mathcal{L}}(x)$ of unary $\mathcal{L}$-terms in the following inductive way:

- the variable x belongs to $C^{\mathcal{L}}(x)$,
- for any k-ary dual operator ∇ of $\mathcal{L}$, any $\blacksquare(x)$ in $C^{\mathcal{L}}(x)$ and any i in $\{1,\ldots,k\}$, the term $\nabla(0,\ldots,0,\blacksquare(x),0,\ldots,0)$ (where only the i^{th} component is not equal to 0) belongs to $C^{\mathcal{L}}(x)$,
- if $\blacksquare_1(x)$ and $\blacksquare_2(x)$ belong to $C^{\mathcal{L}}(x)$ and if m is a positive integer then $\blacksquare_1^m(x)$ and $\blacksquare_1(x)\wedge\blacksquare_2(x)$ belong $C^{\mathcal{L}}(x)$.

If A is an $\mathcal{MMV}^{\mathcal{L}}$-algebra, a non-empty subset F of A is a *modal filter* if F is an implicative filter of the MV-algebra reduct of A and if F contains $\blacksquare(x)$ whenever x belongs to F and $\blacksquare$ belongs to $C^{\mathcal{L}}(x)$.

LEMMA 3.13. *The terms of $C^{\mathcal{L}}(X)$ are interpreted as dual MV-operators on any algebra of $\mathcal{MMV}^{\mathcal{L}}$.*

PROOF. The proof is an easy induction. The only non trivial part is to proof that if $\Box_1$ and $\Box_2$ are two unary dual MV-operators on an MV-algebra A, then $\Box_1\wedge\Box_2$ satisfies (**K**). On the one hand we obtain for any x and y in A,

$$\begin{aligned}(\Box_1\wedge\Box_2)(x)\to(\Box_1\wedge\Box_2)(y) &= (\Box_1x\wedge\Box_2x)\to(\Box_1y\wedge\Box_2y)\\ &= ((\Box_1x\wedge\Box_2x)\to\Box_1y)\wedge((\Box_1x\wedge\Box_2x)\to\Box_2y)\\ &= ((\Box_1x\to\Box_1y)\vee(\Box_2x\to\Box_1y))\\ &\quad\wedge((\Box_1x\to\Box_2y)\vee(\Box_2x\to\Box_2y)).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\Box_1\wedge\Box_2)(x\to y) &= \Box_1(x\to y)\wedge\Box_2(x\to y)\\ &\leq (\Box_1x\to\Box_1y)\wedge(\Box_2x\to\Box_2y)\\ &\leq ((\Box_1x\to\Box_1y)\vee(\Box_2x\to\Box_1y))\\ &\quad\wedge((\Box_1x\to\Box_2y)\vee(\Box_2x\to\Box_2y)).\end{aligned}$$

We have eventually proved that $(\Box_1\wedge\Box_2)(x\to y)\leq(\Box_1\wedge\Box_2)(x)\to(\Box_1\wedge\Box_2)(y)$ □

PROPOSITION 3.14. *Assume that A is a member of $\mathcal{MMV}^{\mathcal{L}}$.*

(1) *If θ is a congruence on A then $1/\theta$ is a modal filter of A.*
(2) *If F is a modal filter of A then the binary relation θ_F defined on A by*

$$(x,y)\in\theta_F \quad \textit{if} \quad (x\to y)\odot(y\to x)\in F$$

is a congruence on A.
(3) *These correspondences provide two isomorphisms between the lattice of congruences of A and the lattice of modal filters of A (ordered by inclusion). Moreover, these isomorphisms are inverse to each other.*

PROOF. The proof is routine. □

3. More about varieties of finitely-valued modal logics

More specific results can be proved for the varieties $\mathcal{MMV}_n^{\mathcal{L}}$ of $\mathbf{K}_n$-algebras and their corresponding logics.

The first result we provide proves that a dual MV-operator on an MV$_n$-algebra A is entirely characterized by its value on $\mathfrak{B}(A)$.

PROPOSITION 3.15. *Assume that $f : A \to B$ and $f' : A \to B$ are two maps between two MV$_n$-algebras A and B. If $f\restriction_{\mathfrak{B}(A)} = f'\restriction_{\mathfrak{B}(A)}$ and if f and f' preserve the terms $\tau_\oplus$ and $\tau_\odot$ then $f = f'$.*

PROOF. Assume that a belongs to A. Then $f(a) = f'(a)$ if and only if for any i in $\{1, \ldots, n\}$

$$\tau_{i/n}(f(a)) = \tau_{i/n}(f'(a)).$$

This equation is equivalent to

$$f(\tau_{i/n}(a)) = f'(\tau_{i/n}(a))$$

since f and f' preserve the term $\tau_{i/n}$. We can conclude the proof since, according to our conventions of Chapter 1 about the terms $\tau_{i/n}$, we can assume that $\tau_{i/n}(a)$ belongs to $\mathfrak{B}(A)$ for any i in $\{1, \ldots, n\}$. $\square$

COROLLARY 3.16. *Any dual MV-operator on an MV$_n$-algebra A is completely characterized by its restriction to $\mathfrak{B}(A)$.*

We just have proved that any dual operator on the idempotent algebra of an MV$_n$-algebra A can be extended in at most one way to an MV-operator an A. The question to determine which of them can actually be extended will be answered in Proposition 5.16, with the help of topological dualities for the varieties $\mathcal{MMV}_n^{\mathcal{L}}$.

Even though we provide a proof of the following result in Chapter 5 (see page 89), it is interesting to state it here.

PROPOSITION 3.17. *If A belongs to $\mathcal{MMV}_n^{\mathcal{L}}$ then the lattice $\mathsf{Con}(A)$ of congruences of A is isomorphic to the lattice $\mathsf{Con}(\mathfrak{B}(A))$ of congruences of $\mathfrak{B}(A)$.*

Properties about congruence lattices in the varieties of boolean algebras with $\mathcal{L}$-operators can thus be translated in the varieties $\mathcal{MMV}_n^{\mathcal{L}}$. Here is a famous example.

THEOREM 3.18. *The variety $\mathcal{MMV}_n^{\mathcal{L}}$ is congruence distributive and congruence permutable.*

The problem of the characterization of subdirectly irreducible algebras in $\mathcal{MMV}_n^{\mathcal{L}}$ can be tackled in different ways. First, we need a construction of the modal filter $\langle X\rangle_\blacksquare$ generated by a subset X of an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra A (which is defined as usual as the intersection of the modal filters of A that contain X).

LEMMA 3.19. *If A is a member of $\mathcal{MMV}_n^{\mathcal{L}}$ and X is a subset of A, then*

$$\langle X\rangle_\blacksquare = \{a \in A \mid a \geq \blacksquare_1 x_1^n \wedge \cdots \wedge \blacksquare_k x_k^n \text{ for some } \blacksquare_1, \ldots, \blacksquare_k \in C^{\mathcal{L}}(x) \text{ and } x_1, \ldots, x_k \in X\}.$$

PROOF. The proof is routine. $\square$

We adapt the definition of an *opremum* (see [**37**] and [**51**]) to our many-valued settings.

DEFINITION 3.20. An element m of an an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra A is an *opremum* if $m \neq 1$ and if for any a in A, there are some $\blacksquare_1, \ldots, \blacksquare_k$ in $C^{\mathcal{L}}(x)$ such that $m \geq \blacksquare_1 a^n \wedge \cdots \wedge \blacksquare_k a^n$.

This definition leads us to a characterization of subdirectly irreducible algebras in $\mathcal{MMV}_n^{\mathcal{L}}$ which is very similar to the characterization that exists for the corresponding boolean algebras with operators (see [**37**], [**51**] and [**57**]).

PROPOSITION 3.21. *Assume that A is a member of $\mathcal{MMV}_n^{\mathcal{L}}$. The following conditions are equivalent.*

(1) *the algebra A is subdirectly irreducible,*
(2) *the boolean algebra with operators $\langle \mathfrak{B}(A), \{\nabla_i \restriction_{\mathfrak{B}(A)^{k_i}} \mid i \in I\}\rangle$ is subdirectly irreducible,*
(3) *the algebra A has an opremum,*
(4) *the algebra $\langle \mathfrak{B}(A), \{\nabla_i \restriction_{\mathfrak{B}(A)^{k_i}} \mid i \in I\}\rangle$ has an opremum.*

PROOF. The equivalence between (1) and (2) is a consequence of Proposition 3.17. Since $\langle \mathfrak{B}(A), \{\nabla_i \restriction_{\mathfrak{B}(A)^{k_i}} \mid i \in I\}\rangle$ is an algebra of $\mathcal{MMV}_n^{\mathcal{L}}$, in order to prove that (2) is equivalent to (4), it suffices to prove that (1) is equivalent to (3).

Now, according to Lemma 3.19, condition (3) means that there is an element m in $A \setminus \{1\}$ which is in the modal filter generated by any element of $A \setminus \{1\}$, hence it is in any non trivial modal filter. Thanks to Proposition 3.14, it means that the intersection of the non trivial congruences of A is not the identity relation. Thus, condition (3) is equivalent to the subirreducibility of A. □

4. Completeness results

We detail in this section several completeness results with respect to relational semantics. The results are obtained thanks to a construction of a *canonical model* and by application of Lemma 2.40. The first result is about the logic $\mathbf{K}_n$ and the Ł$_n$-valued KRIPKE-models.

DEFINITION 3.22. Assume that $\mathbf{L}$ is a many-valued modal $\mathcal{L}$-logic. The *algebraic canonical model of* $\mathbf{L}$ is the algebraic model $\langle \mathcal{F}_{\mathbf{L}}, a_{\bullet}\rangle$ defined by $a_p = p^{\mathbf{L}}$ for any propositional variable p.

The *canonical model of* $\mathbf{L}$ is the canonical KRIPKE-model associated to the algebraic canonical model of $\mathbf{L}$.

4.1. Completeness for $\mathbf{K}_n$.

With the help of the Truth Lemma and the Prime Ideal Theorem for the varieties $\mathcal{MV}_n$, we can derive the completeness of $\mathbf{K}_n$ with respect to the class of the Ł$_n$-valued KRIPKE models.

THEOREM 3.23. *For any positive integer n, an $\mathcal{L}$-formula ϕ is a theorem of $\mathbf{K}_n$ if and only if ϕ is true in any Ł$_n$-valued* KRIPKE *model.*

PROOF. The left to right part of the statement is known. We prove the right to left part. If ϕ is valid in any KRIPKE model of $\mathbf{K}_n$, then it is valid in the canonical model of $\mathbf{K}_n$. Thus, according to Lemma 2.40, the class of ϕ in $\mathcal{F}_{\mathbf{K}_n}$ is in any prime filter of $\mathcal{F}_{\mathbf{K}_n}$, which means that this class is equal to 1. □

Thanks to Theorem 3.23, we can obtain the following simplier axiomatization of $\mathbf{K}_n$.

PROPOSITION 3.24. *Let us denote by $\mathbf{K}'_n$ the smallest set of $\mathcal{L}$-formulas that contains an axiomatization of* ŁUKASIEWICZ $n+1$*-valued logic, the formulas corresponding to the scheme* (K) *of modal logics (as in Definition 3.1), the formulas $\nabla(p_1 \oplus p_1, \dots, p_k \oplus p_k) \leftrightarrow (\nabla(p_1, \dots, p_k) \oplus \nabla(p_1, \dots, p_k))$, $\nabla(p_1 \odot p_1, \dots, p_k \odot p_k) \leftrightarrow (\nabla(p_1, \dots, p_k) \odot \nabla(p_1, \dots, p_k)$ for any dual modality ∇ of $\mathcal{L}$ and that is closed under the uniform substitution rule, the detachment rule and the necessitation rule.*

Then, the logic $\mathbf{K}_n$ is equal to $\mathbf{K}'_n$.

PROOF. In the proof of item (1) of Lemma 2.40, we have never used the fact that a dual MV-operator ∇ satisfies the equation

$$\nabla(x_1 \oplus x_1^m, \dots, x_k \oplus x_k^m) = (\nabla(x_1, \dots, x_k) \oplus (\nabla(x_1, \dots, x_k))^m) \tag{4.1}$$

for any positive integer m.

Thus, Lemma 2.40 is still true in the variety $\{A \mid A \models \mathbf{K}'_n\}$ of the $\mathbf{K}'_n$-algebras. It is now easy to prove that Theorem 3.23 can also be stated if we repalce $\mathbf{K}_n$ by $\mathbf{K}'_n$. Since the formulas from which equations (4.1) are issued are tautologies in $Ł_n$-valued KRIPKE-models, we can deduce that they belong to $\mathbf{K}'_n$ and eventually that $\mathbf{K}_n = \mathbf{K}'_n$. □

Recall Proposition 2.7 in which we list some tautologies for $[0,1]$-valued $\mathcal{L}$-models, which are so theorems of $\mathbf{K}_n$ for any n.

4.2. Completeness for K. Unfortunately, the rules of the HILBERT system associated to $\mathbf{K}$ are too narrow in order to generate in a syntactic way the modal theory of the $[0,1]$-valued models. To obtain such a result, we introduce an infinitary modal system.

DEFINITION 3.25. Assume that ϕ is an $\mathcal{L}$-formula. We note $\vdash_\infty^{\mathcal{L}} \phi$, or simply $\vdash_\infty \phi$, if $\{\phi \oplus \phi^m \mid m \in \mathbb{N}_0\} \subseteq \mathbf{K}^{\mathcal{L}}$.

Obviously, the formula ϕ is such that $\vdash_\infty^{\mathcal{L}} \phi$ if and only if $\vdash_{\mathbf{K}} \phi \oplus \phi^m$ for any non negative integer m.

With this system, we can produce any tautology.

THEOREM 3.26. *If ϕ is a formula of $\mathsf{Form}^{\mathcal{L}}$, then $\vdash_\infty \phi$ if and only if ϕ is true in any $[0,1]$-valued* KRIPKE *model.*

PROOF. If ϕ is an $\mathcal{L}$-formula such that $\vdash_\infty \phi$, then $\vdash_{\mathbf{K}} \phi$ and $\mathcal{MMV}^{\mathcal{L}} \models \phi = 1$ according to Theorem 3.9. Then, if $\mathfrak{F}$ is a frame, Lemma 2.35 states that $\mathfrak{F} \models \phi$ if and only if $\mathfrak{F}^+ \models \phi = 1$. We conclude this part of the proof thanks to Lemma 2.34 from which we derive that $\mathfrak{F}^+$ belongs to $\mathcal{MMV}^{\mathcal{L}}$ so that ϕ is true in any model based on $\mathfrak{F}$.

Conversely, if ϕ is a tautology, then ϕ is true in the canonical model of $\mathbf{K}$. It means that ϕ belongs to any maximal filter of $\mathcal{F}_{\mathbf{K}}$, *i.e.*, that ϕ is an infinitely great element in $\mathcal{F}_{\mathbf{K}}$ and that $\phi \oplus \phi^n$ belongs to $\mathbf{K}$ for any $n \in \mathbb{N}_0$. □

It is still an open problem to determine a minimal extension of $\mathbf{K}$ for which the completeness result with respect to the to KRIPKE $[0,1]$-valued models can be stated.

4.3. Completness for $\mathbf{MPDL}_n$. Let us now provide a set of rules to generate the modal theory of the Ł$_n$-valued models for MPDL. These rules are provided through the smallest Ł$_n$*-valued propositional dynamic logic* $\mathbf{MPDL}_n$. The most original difference between the axiomatization of $\mathbf{MPDL}_n$ and **PDL** appears in the induction axiom and in the axiom that defines the test operator $[\phi?]$.

The proofs of the results about $\mathbf{MPDL}_n$ are usually done by induction on the subexpression relation. The arguments are quite long and not so interesting. We thus avoid to bother the reader with some of these long technical proofs.

DEFINITION 3.27. An Ł$_n$*-valued propositional dynamic logic* (or simply a *logic*) is a subset $\mathbf{L}$ of formulas of Form that is closed under the rules of uniform substitution, detachment and necessitation and that contains the following axioms:

(1) an axiomatic base of the ŁUKASIEWICZ $n+1$-valued logic;
(2) the axioms $[\alpha](p \to q) \to ([\alpha]p \to [\alpha]q)$, $[\alpha](p \oplus p) \leftrightarrow [\alpha]p \oplus [\alpha]p$ and $[\alpha](p \odot p) \leftrightarrow [\alpha]p \odot [\alpha]p$ for any program α of Π;
(3) the axioms that define the program operators: $[\alpha \cup \beta]p \leftrightarrow [\alpha] \wedge [\beta]p$, $[\alpha;\beta]p \leftrightarrow [\alpha][\beta]p$, $[q?]p \leftrightarrow (\neg(q^n) \vee p)$ and $[\alpha^*]p \leftrightarrow (p \wedge [\alpha][\alpha^*]p)$ for any programs α and β of Π;
(4) the induction axiom $p \wedge ([\alpha^*](p \to [\alpha]p)^n) \to [\alpha^*]p$ for any program α.

We denote by $\mathbf{MPDL}_n$ the smallest of the Ł$_n$-valued propositional dynamic logics.

Note that, roughly speaking, the induction axiom means "if after an undetermined number of executions of α the truth value of p cannot decrease after a new execution of α, then the truth value of p cannot decrease after any undetermined number of executions of α". This is a simple generalization of the induction axiom of propositional dynamic logic (which could not have been adopted without modification since it is not a tautology).

Since the axioms of $\mathbf{MPDL}_n$ are tautologies (see Proposition 2.12) and that tautologies are preserved by application of the rules of uniform substitution, detachment and necessitation, we obtain directly the the class of the Ł$_n$-valued KRIPKE models for MPDL forms a sound semantic for $\mathbf{MPDL}_n$.

The problems that arise in proving the completeness result for $\mathbf{MPDL}_n$ are similar to the ones that arise for the completeness result for **PDL**. Indeed, in the construction of the canonical model for $\mathbf{MPDL}_n$, the relation associated to each program are not build inductively from the relation associated to atomic programs. Instead, we directly associate to each α of Π a canonical relation R_α in the canonical way defined in Definition 2.37. In fact, the inductive rules involving the operators ";", "$\cup$" and "? " are satisfied in the canonical model, but R_{α^*} is greater than the transitive and reflexive closure of R_α. We use the technique of filtration to construct an Ł$_n$-valued KRIPKE model from this canonical model.

DEFINITION 3.28. A *non standard* Ł$_n$*-valued* KRIPKE *model* $\mathcal{M} = \langle W, R_{\bullet}, \mathrm{Val}\rangle$ *for MPDL*$_n$ is given by an non empty set W, a map $R : \Pi \to 2^{W \times W}$ and a map $\mathrm{Val} : W \times \mathsf{Prop} \to$ Ł$_n$ (which is extended to formulas in the usual way) such that for any program α and β and any formula ψ the identities $R_{\alpha;\beta} = R_\alpha \circ R_\beta$, $R_\alpha \cup R_\beta$ and $R_{\psi?} = \{(u,u) \mid \mathrm{Val}(u,\psi) = 1\}$ are satisfied and such that R^*_α is a transitive and reflexive extension of R_α.

We use the FISHER - LADNER closure map $\mathrm{FL} : \mathsf{Form} \to 2^{\mathsf{Form}}$ to prove a filtration lemma for non standard models. This closure map is introduced in order to deal with the interdefinability of programs and formulas in proofs that use induction on the subexpression relation. We follow section 6.1 of [**31**] to introduce this map, since there is nothing really new in this syntactic aspect of $\mathbf{MPDL}_n$.

DEFINITION 3.29. The FISHER - LADNER *closure* $\mathrm{FL}(\phi)$ *of a formula* ϕ is defined inductively with the help of the map $\mathrm{FL}^{\Box} : \{[\alpha]\phi \mid \alpha \in \Pi, \phi \in \mathsf{Form}\} \to 2^{\mathsf{Form}}$ which is defined by simultaneous induction with FL by the following rules:

(1) $\mathrm{FL}(p) = \{p\}$ for any propositional variable p,
(2) $\mathrm{FL}(\psi \oplus \phi) = \{\psi \oplus \phi\} \cup \mathrm{FL}(\phi) \cup \mathrm{FL}(\psi)$,
(3) $\mathrm{FL}([\alpha]\phi) = \mathrm{FL}^{\Box}([\alpha]\phi) \cup \mathrm{FL}(\phi)$,
(4) $\mathrm{FL}^{\Box}([a]\phi) = \{[a]\phi\}$ for any atomic program a,
(5) $\mathrm{FL}^{\Box}([\alpha \cup \beta]\phi) = \{[\alpha \cup \beta]\phi\} \cup \mathrm{FL}^{\Box}([\alpha]\phi) \cup \mathrm{FL}^{\Box}([\alpha]\phi)$,
(6) $\mathrm{FL}^{\Box}([\alpha;\beta]\phi) = \{[\alpha;\beta]\phi\} \cup \mathrm{FL}^{\Box}([\alpha][\beta]\phi) \cup \mathrm{FL}^{\Box}([\beta]\phi)$,
(7) $\mathrm{FL}^{\Box}([\alpha^*]\phi) = \{[\alpha^*]\phi\} \cup \mathrm{FL}^{\Box}([\alpha][\alpha^*]\phi)$,
(8) $\mathrm{FL}^{\Box}([\psi?]\phi) = \{[\psi?]\phi\} \cup \mathrm{FL}(\psi)$.

The following justifies the word "closure" in "FISHER - LADNER closure."

LEMMA 3.30. *Assume that* ϕ *belongs to* Form.

(1) *If* ψ *belongs to* $\mathrm{FL}(\phi)$ *then* $\mathrm{FL}(\psi) \subseteq \mathrm{FL}(\phi)$.
(2) *If* ψ *belongs to* $\mathrm{FL}^{\Box}([\alpha]\phi)$ *then* $\mathrm{FL}(\psi) \subseteq \mathrm{FL}^{\Box}([\alpha]\phi) \cup \mathrm{FL}(\phi)$.

PROOF. The proof is done by simultaneous induction on the subexpression relation. See Lemma 6.1 in [**31**]. □

We can now turn to filtrations of non standard models.

DEFINITION 3.31. If $\mathcal{M} = \langle W, R_{\bullet}, \mathrm{Val}\rangle$ is a non standard Ł$_n$-valued KRIPKE model for MPDL, we define the equivalence relation $\equiv_\phi$ on W for any ϕ in Form by

$$u \equiv_\phi v \quad \text{if} \quad \forall\psi \in \mathrm{FL}(\phi)\ \mathrm{Val}(u,\phi) = \mathrm{Val}(v,\phi).$$

For the sake of readability, we denote by $[W]_\phi$ (or simply by $[W]$) the quotient of W by $\equiv_\phi$ and by $[v]_\phi$ (or simply by $[v]$) the class of an element v of W for $\equiv_\phi$.

Then, for any atomic program a of Π_0 we define $R_a^{[\mathcal{M}]}$ as the binary relation on $[W]$ that collects the $([u],[v])$ such that

$$[u] \times [v] \cap R_a \neq \emptyset$$

and the map $\mathrm{Val}^{[W]}$ on $[W] \times \mathsf{Prop}$ by

$$\mathrm{Val}^{[\mathcal{M}]}([u],p) = \max\{\frac{i}{n} \in \text{Ł}_n \mid \mathrm{Val}(\cdot,p)^{-1}(\frac{i}{n}) \cap [u] \neq \emptyset\}.$$

The model $[\mathcal{M}]_\phi = \langle [W]_\phi, R^{[\mathcal{M}]_\phi}, \mathrm{Val}^{[\mathcal{M}]_\phi}\rangle$ (or simply $[\mathcal{M}] = \langle [W], R^{[\mathcal{M}]}, \mathrm{Val}^{[\mathcal{M}]}\rangle$) is *the filtration of* $\mathcal{M}$ *through* ϕ.

LEMMA 3.32 (Filtration). *Assume that* $\mathcal{M} = \langle W, R, \mathrm{Val}\rangle$ *is a non standard model and* ϕ *is a formula.*

(1) *Assume that* ψ *is in* $\mathrm{FL}(\phi)$ *and that* i *is in* $\{0,\ldots,n\}$. *Then* $\mathrm{Val}(u,\psi) \geq \frac{i}{n}$ *if and only if* $\mathrm{Val}^{[W]}([u],\psi) \geq \frac{i}{n}$.
(2) *For every* $[\alpha]\psi$ *in* $\mathrm{FL}(\phi)$,

(a) *if* $(u,v) \in R_\alpha$ *then* $([u],[v]) \in R_\alpha^{[\mathcal{M}]}$;
(b) *if* $i \in \{0,\dots,n\}$, *if* $([u],[v]) \in R_\alpha^{[\mathcal{M}]}$ *and if* $\mathrm{Val}(u,[\alpha]\psi) \geq \frac{i}{n}$ *then* $\mathrm{Val}(v,\psi) \geq \frac{i}{n}$.

PROOF. The proof is a long but not so hard argument by induction on the subexpression relation. We leave it to the reader. □

Let us define the canonical model of $\mathbf{MPDL}_n$. This construction uses the LINDENBAUM - TARSKI algebra of $\mathbf{MPDL}_n$.

DEFINITION 3.33. We denote by $\mathcal{F}_{\mathbf{MPLD}_n}$ the LINDENBAUM - TARSKI *algebra of* $\mathbf{MPDL}_n$, *i.e.*, the quotient of Form by the relation of syntactical equivalence $\equiv$ ($\phi \equiv \psi$ if $\phi \leftrightarrow \psi \in \mathbf{MPDL}_n$).

Since the reduct of $\mathcal{F}_{\mathbf{MPLD}_n}$ to the language of MV-algebras is an MV_n-algebra, we can adopt the following definition.

DEFINITION 3.34. The *canonical model of* $\mathbf{MPDL}_n$ is the model

$$\mathcal{M}_{\mathbf{MPDL}_n} = \langle W^{\mathbf{MPDL}_n}, R^{\mathbf{MPDL}_n}, \mathrm{Val}^{\mathbf{MPDL}_n}\rangle$$

where

- $W^{\mathbf{MPDL}_n} = \mathcal{MV}(\mathcal{F}_{\mathbf{MPDL}_n}, \text{Ł}_n)$;
- for any program α, the relation $R_\alpha^{\mathbf{MPDL}_n}$ collects the (u,v) such that

$$\forall x \in \mathcal{F}_{\mathbf{MPDL}_n}\ (u([\alpha]x) = 1 \Rightarrow v(x) = 1);$$

- the map $\mathrm{Val}^{\mathbf{MPDL}_n}$ is defined by $\mathrm{Val}^{\mathbf{MPDL}_n}(u,p) = u(p)$ for any propositional variable p in Prop.

When no confusion is possible, we prefer to write R and Val instead of $R^{\mathbf{MPDL}_n}$ and $\mathrm{Val}^{\mathbf{MPDL}_n}$.

The major result is the following one.

THEOREM 3.35. *The canonical model of* $\mathbf{MPDL}_n$ *is a non standard* KRIPKE *model. Moreover, for any formula* ϕ *and any element* u *of* $W^{\mathbf{MPDL}_n}$,

$$\mathrm{Val}(u,\psi) = u(\psi/\equiv).$$

PROOF. The proof is once again an induction argument. □

This model provides the desired completeness result.

THEOREM 3.36. *An* $\mathcal{L}_{\Pi_0}$*-formula is a theorem of* $\mathbf{MPDL}_n$ *if and only if it is a tautology.*

PROOF. Only the completeness part requires a proof. Assume that ϕ is a tautology. Then, the formula ϕ is true in the filtration of the canonical model of $\mathbf{MPDL}_n$ through ϕ, hence in the canonical model of $\mathbf{MPDL}_n$ thanks to Lemma 3.32. □

4.4. Canonical logics and canonical varieties. Up to now, we have considered the question of completeness relatively to classes of models (and varieties of algebras). We here develop algebraic tools to tackle the question of completeness relatively to classes of *structures*. Once again, the result we obtain are about finitely valued logics. The reader should so keep in mind that we fix a positive integer n and consider $Ł_n$-valued logics and $Ł_n$-valued valuation for the end of the dissertation.

DEFINITION 3.37. An $Ł_n$-valued modal $\mathcal{L}$-logic $\mathbf{L}$ is KRIPKE *complete* if there is a class K of $Ł_n$-valued $\mathcal{L}$-frames such that $\mathbf{L} = \{\phi \in \mathsf{Form} \mid \forall \mathfrak{F} \in K, \mathfrak{F} \models \phi\}$.

The logic $\mathbf{L}$ is *strongly* KRIPKE *complete* if there is a class K of $\mathcal{L}$-frames such that $\mathbf{L} = \{\phi \in \mathsf{Form} \mid \forall \mathfrak{F} \in K, \mathfrak{F} \models_n \phi\}$.

We have a few remarks about these definitions. First note that in the terms "KRIPKE complete" and "strongly KRIPKE complete", there is no explicit reference on n, even tough the notions they define are dependent of n. This improper use of the vocabulary doe not jeopardize the understanding of the results since we assume that the integer n is fixed one for all for the end of the dissertation.

Furthermore, one who is used to classical modal logic would probably have called by KRIPKE *complete logic* what we call a *strongly* KRIPKE *complete logic*, *i.e.*, a logic which is complete with respect to a class of $\mathcal{L}$-frames. But then, one would have lost the connection between canonical logics and KRIPKE complete logics. Thus, our choice of vocabulary is done in a way that helps to compare our results with classical ones.

Finally, note that our definition of *strong* KRIPKE *completeness* has nothing to do with the classical definition of a *strongly complete logic with respect to a class K of structures* (a logic $\mathbf{L}$ such that any formula ψ that is a local semantic consequence of a set of formulas Φ in the class K of structures is $\mathbf{L}$-deducible from Φ). Our choice is justified by the fact that the notion of a *strongly* KRIPKE *complete logic* is definitely *stronger* than the notion of a KRIPKE complete logic. Indeed, if $\mathbf{L}$ is an $Ł_n$-valued logic and if K is a class of $\mathcal{L}$-frames such that

$$\mathbf{L} = \{\phi \in \mathsf{Form} \mid \forall \mathfrak{F} \in K, \mathfrak{F} \models_n \phi\},$$

then it follows obviously that if K' denotes the class of the trivial $Ł_n$-valued $\mathcal{L}$-frames based on the $\mathcal{L}$-frames of K,

$$\mathbf{L} = \{\phi \in \mathsf{Form}^{\mathcal{L}} \mid \forall \mathfrak{F} \in K', \mathfrak{F} \models \phi\}.$$

Moreover, the following example proves that there exists a logic that is strongly KRIPKE complete without being KRIPKE complete.

EXAMPLE 3.38. The logic $\mathbf{L} = \mathbf{K}_n + \Box(p \vee \neg p)$ is KRIPKE complete but is not strongly KRIPKE complete.

We prove the completeness part in two steps. First, we prove that, the fact that $\Box(p \vee \neg p)$ is true in any algebraic model $\langle \mathcal{F}_{\mathbf{L}}, a_{\cdot} \rangle$ based on $\mathcal{F}_{\mathbf{L}}$ forces the canonical frame of $\mathbf{L}$ to satisfy the first order formula $\forall u(Ru \subseteq r_1)$. Then, we prove that if an $Ł_n$-valued $\mathcal{L}$-frame satisfies this first order formula, then it validates $\Box(p \vee \neg p)$.

Let us prove that $\mathbf{L}$ is not strongly KRIPKE complete. Proceed *ad absurdum* and assume that K is a class of frames such that $\mathbf{L} = \{\phi \mid K \models_n \phi\}$. Then, K contains a frame whose accessibility relation is not empty. Otherwise, the formula $\Box\phi$ belongs to $\mathbf{L}$ for any ϕ, while $\Box(p \wedge \neg p)$ does not belong to $\mathbf{L}$.

So, let us denote by $\mathfrak{F}$ such a frame, by $\mathcal{M} = \langle W, R, \mathrm{Val}\rangle$ a model based on $\mathfrak{F}$ and by u, v two elements of W such that $(u, v) \in R$. Since $\mathcal{M}, w \models \Box(p \vee \neg p)$, it follows that $\mathrm{Val}(p, v) \in \{0, 1\}$. Then, if we denote by $\mathcal{M}' = \langle W, R, \mathrm{Val}'\rangle$ the model based on $\mathfrak{F}$ defined by

$$\mathrm{Val}'(q, u) = \begin{cases} \mathrm{Val}(q, u) \text{ if } q \neq p \text{ or } u \neq v, \\ \frac{1}{n} \text{ if } q = p \text{ and } u = v, \end{cases}$$

it appears that $\Box(p \vee \neg p)$ is not true in $\mathcal{M}'$, which is the desired contradiction.

These notions of completeness have algebraic translations.

DEFINITION 3.39. A variety $\mathcal{A}$ of MV_n-algebras with $\mathcal{L}$-operators is *complete* if there is a class K of $Ł_n$-valued $\mathcal{L}$-frames such that $\mathcal{A} = \mathbf{Var}_n(K)$.

The variety $\mathcal{A}$ is *strongly complete* if there is a class K of $\mathcal{L}$-frames such that $\mathcal{A} = \mathbf{Var}(K)$.

Of course, a variety $\mathcal{A}$ is complete if and only if $\mathcal{A} = \mathbf{Var}(\mathbf{Str}_n\mathcal{A})$, *i.e.*, if and only if $\mathcal{A}$ is generated by its $Ł_n$-tight complex algebras. It is strongly complete if and only if $\mathcal{A} = \mathbf{Var}(\mathbf{Str}\mathcal{A})$, *i.e.*, if and only if $\mathcal{A}$ is generated by its $Ł_n$-valued complex algebras.

Once again, a strongly complete variety is a complete variety (since the $Ł_n$-tight complex algebra of an $Ł_n$-valued $\mathcal{L}$-frame is a subalgebra of the $Ł_n$-valued complex algebra of its underlying $\mathcal{L}$-frame).

PROPOSITION 3.40. *Assume that* $\mathbf{L}$ *is an* $Ł_n$*-valued modal* $\mathcal{L}$*-logic.*

(1) *The logic* $\mathbf{L}$ *is* KRIPKE *complete if and only if the variety of* $\mathbf{L}$*-algebras is complete.*
(2) *The logic* $\mathbf{L}$ *is strongly* KRIPKE *complete if and only if the variety of* $\mathbf{L}$*-algebras is strongly complete.*

PROOF. (1) Assume that $\mathbf{L} = \bigcap\{\{\phi \in \mathsf{Form} \mid \mathfrak{F} \models \phi\} \mid \mathfrak{F} \in K\}$ for a class K of $Ł_n$-valued $\mathcal{L}$-frames. Then, the variety $\mathcal{MMV}^{\mathcal{L}}_{\mathbf{L}}$ of $\mathbf{L}$-algebras is the variety of the algebras that satisfy the equation that are valid in $\mathfrak{F}^{\times_n}$ for every $\mathfrak{F}$ in K. Equivalently, the variety $\mathcal{MMV}^{\mathcal{L}}_{\mathbf{L}}$ is generated by K.

The proof of (2) is similar. $\Box$

One way to obtain KRIPKE completeness results is through canonicity.

DEFINITION 3.41. An $Ł_n$-valued modal $\mathcal{L}$-logic $\mathbf{L}$ is *canonical* if $\mathbf{L}$ is valid in the canonical $Ł_n$-valued $\mathcal{L}$-frame associated to $\mathcal{F}_{\mathbf{L}}(X)$ for any set X. The logic $\mathbf{L}$ is *strongly canonical* if $\mathbf{L}$ is valid in the canonical $\mathcal{L}$-frame associated to $\mathcal{F}_{\mathbf{L}}(X)$ for any set X.

Any canonical logic $\mathbf{L}$ is KRIPKE-complete. Indeed, thanks to Proposition 4.31, in that case, the logic $\mathbf{L}$ coincides with the set of formulas that are valid in the canonical $Ł_n$-valued $\mathcal{L}$-frame of $\mathcal{F}_{\mathbf{L}}(\omega)$. The same line of argument can be used to prove that any strongly canonical logic $\mathbf{L}$ is strongly KRIPKE complete.

Determining which logics are canonical or strongly canonical is so a interesting problem. This problem is associated to an algebraic one.

DEFINITION 3.42. A variety $\mathcal{A}$ of $\mathcal{MMV}^{\mathcal{L}}_n$-algebras is *canonical* if A contains the canonical extension of its members.

A variety $\mathcal{A}$ of $\mathcal{MMV}^{\mathcal{L}}_n$-algebras is *strongly canonical* if A contains the algebra $(A_{+_n})^{+_n}$ for any A in $\mathcal{A}$.

Corollary 4.32 proves that this definition of canonicity coincides with the definition of the canonicity for a class of expanded bounded distributive lattices. Furthermore, in the sequel, we prove that an Ł_n-valued $\mathcal{L}$-logic $\mathbf{L}$ is (strongly) canonical if and only if the variety of $\mathbf{L}$-algebras is (strongly) canonical (see Proposition 4.33 and Proposition 4.59). Obtaining tools that help to generate (strongly) canonical varieties is so an interesting problem.

We use in the sequel the notations introduced in Definition 2.73. Let us also recall that the Ł_n-valued canonical extension has been defined for $\mathcal{L}$-frames *and* Ł_n-valued $\mathcal{L}$-frames (see Definition 2.63).

Theorem 3.43. *Assume that K is a class of structures.*

1. *If K is a class of Ł_n-valued $\mathcal{L}$-frames then $\mathbf{Var}_n(K)$ is a canonical variety if and only if $\mathbf{Str}_n\mathbf{Var}_n(K)$ is closed under Ł_n-valued canonical extensions.*
2. *If K is a class of $\mathcal{L}$-frames then $\mathbf{Var}(K)$ is a strongly canonical variety if and only if $\mathbf{StrVar}(K)$ is closed under Ł_n-valued canonical extensions.*

Proof. Both results can be proved in parallel. We adopt the following notations

Notation	Substitution rule for (1)	Substitution rule for (2)
Cm	$\mathbf{Cm}_n$	$\mathbf{Cm}$
Var	$\mathbf{Var}_n$	$\mathbf{Var}$
$\mathfrak{F}^{\mathsf{e}}$	$\mathfrak{F}^{\mathsf{me}_n}$	$\mathfrak{F}^{\mathsf{me}}$
$\mathfrak{F}^{\times}$	$\mathfrak{F}^{\times_n}$	$\mathfrak{F}^{+_n}$
$A_{\times}$	$A_{\times_n}$	A_{+_n}
Str	$\mathbf{Str}_n$	$\mathbf{Str}$
canonical (about varieties)	canonical	strongly canonical.

To obtain a proof of (1) (resp. of (2)), we substitute in the sequel any occurrence of an element of the first column of the previous array by its translation in the second column (resp. in the third column).

First assume that $\mathrm{Var}(K)$ is canonical and that $\mathfrak{F}$ belongs to $\mathsf{Str}\mathrm{Var}(K)$. By definition, the canonical extension $\mathfrak{F}^{\mathsf{e}}$ of $\mathfrak{F}$ is the structure $(\mathfrak{F}^{\times})_{\times}$ where $\mathfrak{F}^{\times}$ belongs to $\mathrm{Var}(K)$. Since $\mathrm{Var}(K)$ is a canonical variety, we obtain that $((\mathfrak{F}^{\times})_{\times})^{\times}$ still belongs to $\mathrm{Var}(K)$ which means that $\mathfrak{F}^{\mathsf{e}} = (\mathfrak{F}^{\times})_{\times}$ belongs to $\mathsf{Str}\mathrm{Var}(K)$.

Conversely, assume that $\mathsf{Str}\mathrm{Var}(K)$ is closed under Ł_n-valued canonical extensions and that A is a member of $\mathrm{Var}(K) = \mathbb{HSP}\mathrm{Cm}(K)$. Then, there is a B in $\mathbb{SP}\mathrm{Cm}(K)$ such that A is an homomorphic image of B. Thanks to Proposition 2.49, there is a family $\{\mathfrak{F}_j \mid j \in J\}$ of structures of K such that B appears as a subalgebra of $(\mathfrak{F})^{\times} = (\biguplus\{\mathfrak{F}_j \mid j \in J\})^{\times}$. By dualizing the previous arrows, we obtain

$$(A_{\times})^{\times} \twoheadleftarrow (B_{\times})^{\times} \hookrightarrow ((\mathfrak{F}^{\times})_{\times})^{\times}. \tag{4.2}$$

Since $\mathfrak{F}^{\times} = \prod\{\mathfrak{F}_j^{\times} \mid j \in J\}$ belongs to $\mathrm{Var}(K)$, the structure $\mathfrak{F}$ is a member of $\mathsf{Str}\mathrm{Var}(K)$. So, since $\mathsf{Str}\mathrm{Var}(K)$ is closed for canonical extensions, we obtain that $\mathfrak{F}^{\mathsf{e}} = (\mathfrak{F}^{\times})_{\times}$ belongs to $\mathsf{Str}\mathrm{Var}(K)$ and eventually that $((\mathfrak{F}^{\times})_{\times})^{\times}$ is a member of $\mathrm{Var}(K)$, which concludes the proof thanks to the arrows of (4.2). □

The next result, which is again a simple adaptation of a result of Goldblatt, (see **[25]**) requires the following lemma.

LEMMA 3.44. *If* $(\mathsf{Cst}, \mathsf{H}, \mathsf{S}, \mathrm{Cm}, \mathsf{Ex}) \in \{(\mathbf{Cst}_n, \mathbf{H}_n, \mathbf{S}_n, \mathbf{Cm}_n, \mathbf{Ex}_n), (\mathbf{Cst}, \mathbf{H}, \mathbf{S}, \mathbf{Cm}, \mathbf{Ex})\}$, *then* $\mathsf{Cst}\mathbb{H}\mathbb{S} \leq \mathsf{SHCst}$ *and* $\mathsf{Cst}\mathbb{H}\mathbb{S}\mathrm{Cm} \leq \mathsf{SHEx}$.

PROOF. The first inequality follows by dualization of arrows and the second is a consequence of the first. □

THEOREM 3.45. *Assume that $\mathcal{A}$ is a variety of MV_n-algebras with $\mathcal{L}$-operators.*

(1) *The variety $\mathcal{A}$ is canonical if and only if it is complete and the class $\mathbf{Str}_n(\mathcal{A})$ is closed under Ł$_n$-valued canonical extensions.*
(2) *The variety $\mathcal{A}$ is strongly canonical if and only if it is strongly complete and the class $\mathbf{Str}(\mathcal{A})$ is closed under Ł$_n$-valued canonical extensions.*

PROOF. Once again, the two results can be proved in parallel. For the notations, we adopt the same conventions as in the proof of Theorem 3.43 and we add the following rules to our substitution guide :

Notation	Substitution rule for (1)	Substitution rule for (2)
complete	complete	strongly complete
complex	Ł$_n$-tight complex	Ł$_n$-complex
H	$\mathbf{H}_n$	$\mathbf{H}$
S	$\mathbf{S}_n$	$\mathbf{S}$
Ex	$\mathbf{Ex}_n$	$\mathbf{Ex}$.

We already know that a canonical variety is a complete variety. So, conversely, let us assume that $\mathcal{A}$ is a complete variety and that the class $\mathsf{Str}(\mathcal{A})$ is closed under canonical extensions.

We obtain successively

$$\begin{aligned} \mathcal{A} &= \mathbb{H}\mathbb{S}\mathbb{P}\mathrm{Cm}\mathsf{Str}(\mathcal{A}) \\ &= \mathbb{H}\mathbb{S}\mathrm{Cm}\mathbf{Ud}\mathsf{Str}(\mathcal{A}) \\ &= \mathbb{H}\mathbb{S}\mathrm{Cm}\mathsf{Str}(\mathcal{A}) \end{aligned}$$

where the last identity is obtained thanks to the fact that $\mathcal{A}$ is closed under $\mathbb{P}$. It follows that

$$\mathsf{Cst}(\mathcal{A}) = \mathsf{Cst}\mathbb{H}\mathbb{S}\mathrm{Cm}\mathsf{Str}(\mathcal{A}) \subseteq \mathsf{SHExStr}(\mathcal{A}) \subseteq \mathsf{SHStr}(\mathcal{A}) \subseteq \mathsf{Str}(\mathcal{A}),$$

where the first inclusion is obtained by Lemma 3.44, the second by the fact that $\mathsf{Str}(\mathcal{A})$ is closed under canonical extensions and the third by using the property of closure of $\mathcal{A}$ under $\mathbb{H}$ and $\mathbb{S}$.

Hence, if A is an algebra of $\mathcal{A}$, then its associated structure $A_\times$ belongs to $\mathsf{Str}(\mathcal{A})$ which means that $(A_\times)^\times$ belongs to $\mathcal{A}$. □

LEMMA 3.46. *If* $(\mathsf{Cst}, \mathrm{Var}, \mathsf{S}, \mathsf{H})$ *belongs to*

$$\{(\mathbf{Cst}_n, \mathbf{Var}_n, \mathbf{S}_n, \mathbf{H}_n), (\mathbf{Cst}, \mathbf{Var}, \mathbf{S}, \mathbf{H})\}$$

then $\mathsf{Cst}\mathrm{Var} \leq \mathsf{SH}\mathbf{UdPu}$.

PROOF. It suffices to prove that

$$\mathbf{PwUd} \leq \mathsf{H}\mathbf{UdPu}. \tag{4.3}$$

Indeed, we have proved in Theorem 2.75 that

$$\mathbf{Cst}_n\mathbf{Var}_n \leq \mathbf{S}_n\mathbf{H}_n\mathbf{PwUd}.$$

The proof can be mimic in order to obtain

$$\mathsf{Cst}\mathrm{Var} \leq \mathsf{SH}\mathbf{PwUd}. \tag{4.4}$$

Thanks to (4.3), it then follows that $\mathsf{Cst}\mathrm{Var} \leq \mathsf{SH}\mathbf{UdPu}$, which is the desired result.

To prove that inequality (4.3) is true, it suffices to prove that $\mathbf{Pu}\mathsf{Ub} \leq \mathsf{Ub}\mathbf{Pu}$. Indeed, it then follows that

$$\mathbf{PwUd} \leq \mathbf{PwUb} \leq \mathbf{PuUb} \leq \mathbf{UbPu} \leq \mathsf{H}\mathbf{UdPu}$$

thanks to Lemma 2.29.

So, let us prove that $\mathbf{Pu}\mathsf{Ub} \leq \mathsf{Ub}\mathbf{Pu}$. Assume that $\mathfrak{F}$ belongs to $\mathbf{Pu}\mathsf{Ub}$. Then $\mathfrak{F}$ is an ultraproduct $\prod_{j\in J}\mathfrak{F}_j/F$ where for any j in J the structure $\mathfrak{F}_j$ is the bounded union $\cup\{\mathfrak{F}_{i_j} \mid i_j \in I_j\}$ of some structures $\mathfrak{F}_{i_j}$ (where i_j belongs to I_j) of K.

We are going to prove that for any w in $\mathfrak{F}$ there is a generated substructure $\mathfrak{F}_w$ of $\mathfrak{F}$ that contains w and that belongs to $\mathbf{Pu}(K)$. Let w be the element $(w_j)_{j\in J}/F$ of $\mathfrak{F}$. Then, since for any j in J the structure $\mathfrak{F}_j$ is the bounded union for i_j in I_j of the $\mathfrak{F}_{i_j}$'s, it follows that there is a generated substructure $\mathfrak{F}_{i_{jw}}$ with $i_{jw} \in I_j$ that contains w_j. We denote by $\mathfrak{F}_w$ the structure $\prod_{j\in J}\mathfrak{F}_{i_{jw}}/F$ that belongs to $\mathbf{Pu}(K)$. We prove that $\mathfrak{F}_w$ is a generated substructure of $\mathfrak{F}$. Indeed, if

$$\phi : \prod_{j\in J}\mathfrak{F}_{i_{jw}} \to \prod_{j\in J}\mathfrak{F}_j$$

denotes the natural inclusion map, then we can easily prove that the induced map

$$\psi : \prod_{j\in J}\mathfrak{F}_{i_{jw}}/F \to \prod_{j\in J}\mathfrak{F}_j/F : (u_j)_{j\in J}/F \mapsto \phi((u_j)_{j\in J})/F$$

is one-to-one.

Let us prove that ψ is a bounded morphism. Assume that

$$((u_j)_{j\in J}/F, (v_{1j})_{j\in J}/F, \ldots, (v_{kj})_{j\in J}/F) \in R^{\mathfrak{F}_w}.$$

Then, F contains

$$\{j \in J \mid (u_j, v_{1j}, \ldots, v_{kj}) \in R^{\mathfrak{F}_{i_{jw}}}\}$$

which is equal to

$$\{j \in J \mid (u_j, v_{1j}, \ldots, v_{kj}) \in R^{\mathfrak{F}_j}\}$$

since $\mathfrak{F}_{i_{jw}}$ is a generated substructure of $\mathfrak{F}_j$ for any j in J. This last set is equal to

$$\{j \mid ((\phi((u_j)_{j\in J}))_j, (\phi((v_{1j})_{j\in J}))_j, \ldots, (\phi((v_{kj})_{j\in J}))_j) \in R^{\mathfrak{F}_j}\}$$

and this identity allows to conclude that

$$(\psi((u_j)_{j\in J}/F), \psi((v_{1j})_{j\in J}/F), \ldots, \psi((v_{kj})_{j\in J}/F)) \in R^{\mathfrak{F}}$$

Assume now that

$$(\psi((u_j)_{j\in J}/F)), (v'_{1j})_{j\in J}/F, \ldots (v'_{kj})_{j\in J}/F) \in R^{\mathfrak{F}}.$$

Since for any j in J the element $(\psi((u_j)_{j\in J})))_j = u_j$, we obtain that, if j is in J and if $(\phi((u_j)_{j\in J}))_j, v'_{1j}, \dots v'_{kj})$ belongs to $R^{\mathfrak{F}_j}$, then

$$(\phi((u_j)_{j\in J}))_j, v'_{1j}, \dots, v'_{kj}) \in R^{\mathfrak{F}_{i_j w}}$$

since $\mathfrak{F}_{i_j w}$ is a generated substructure of $\mathfrak{F}_j$. This leads us to the desired result.

We eventually prove easily that $\psi(r_m^{\mathfrak{F}_w}) \subseteq r_m^{\mathfrak{F}}$ and that $\psi^{-1}(r_m^{\mathfrak{F}}) \subseteq r_m^{\mathfrak{F}_w}$ for any divisor m of n. □

THEOREM 3.47. *Assume that $\mathcal{A}$ is a variety of MV_n-algebras with $\mathcal{L}$-operators.*

(1) *If the variety $\mathcal{A}$ is generated by a elementary class of Ł$_n$-valued $\mathcal{L}$-frames then it is a canonical variety.*
(2) *If the variety $\mathcal{A}$ is generated by an elementary class of $\mathcal{L}$-frames, then it is a strongly canonical variety.*

PROOF. Both proofs can be made in parallel. So assume that K is an elementary class of structures. Then, thanks to the previous lemma, we obtain

$$\mathsf{Cst}\mathrm{Var}(K) \subseteq \mathsf{SH}\mathbf{UdPu}(K) = \mathsf{SH}\mathbf{Ud}(K) \subseteq \mathsf{Str}\mathrm{Var}(K)$$

since $\mathrm{Var}(K)$ is closed under $\mathbb{P}$, $\mathbb{S}$ and $\mathbb{H}$. Hence, if A belongs to $\mathrm{Var}(K)$, then $(A_\times)^\times$ belongs to $\mathrm{Cm}\mathsf{Cst}\mathrm{Var}(K)$ and so to $\mathrm{Cm}\mathsf{Str}\mathrm{Var}(K)$ and finally, by definition, to $\mathrm{Var}(K)$. □

CHAPTER 4

Canonicity in $\mathcal{MMV}_n^{\mathcal{L}}$: a syntactic approach

It is the leitmotiv of algebraic logic to try to obtain results about logical systems by studying some of their algebraic counterparts. We have illustrated this technique in the previous chapters (completeness results are obtained thanks to canonical constructions based on LINDENBAUM - TARKSI algebras, modally definable classes of structures are obtained through complex algebras *etc.*)

We have for example emphasized the interest of producing (strong) canonical varieties. We indeed know that if a variety associated to a logic **L** is (strongly) canonical, then so is the logic. Since (strongly) canonical logics are (strongly) KRIPKE complete, the algebraic problem of (strong) canonicity is deeply related to the logical problem of (strong) KRIPKE completeness.

In this chapter, we approach the problem of canonicity in a syntactic way. Our goal is to determine classes of equations that define canonical varieties. Because the variety of MV-algebras is not canonical (only its finitely generated subvarieties are, see [**16**]), the presented results are about $Ł_n$-valued modal logics and their varieties.

This famous approach was initiated by JÓNSSON and TARSKI in their seminal work [**33**] and [**34**] about canonical extension of boolean algebras with operators. Their work was later extended to bounded distributive lattices with operators (see [**18**]), bounded distributive lattices with monotone maps (see [**19**]), bounded distributive expansions (see [**20**]) and finally to bounded lattices (see [**17**]).

We first recall the definitions and some usefsull results about the theory of canonical extensions of bounded distributive lattice expansions. We then apply this theory to subvarieties of $\mathcal{MMV}_n$ and obtain the many-valued counterpart of the SAHLQVIST theorem about canonical equations (Theorem 4.41 and 4.61).

Canonical properties are related to classes of $Ł_n$-valued $\mathcal{L}$-frames. In order to obtain completeness results about $\mathcal{L}$-frames, we need an other type of extension. We call that extension the *strong canonical extension*. The last section of the chapter is devoted to that type of extension and we obtain a SAHLQVIST equivalent.

1. Canonical extensions of bounded distributive lattice expansions

We first introduce in a succinct way the theory of canonical extensions for expanded bounded distributive lattices. Our presentation follows the lines of the paper [**20**]. So, we refer to [**20**] for the proofs of the results that we present. We denote by $\mathcal{DL}$ the variety of bounded distributive lattices and we sometimes write "A is a $\mathcal{DL}$" instead of "A belongs to $\mathcal{DL}$". Moreover, every lattice that is considered in this dissertation is bounded. So, in the sequel, by *lattice* we always *bounded lattice*.

DEFINITION 4.1. A complete lattice A is *doubly algebraic* if it is algebraic and if its order dual A^α is algebraic. If A is a complete lattice, we denote respectively by $J^\infty(A)$ and $M^\infty(A)$ the set of the completely join irreducible elements of A and the set of the completely meet irreducible elements of A. The set of the finite joins of elements of $J^\infty(A)$ is denoted by $J^\infty_\omega(A)$ and the set of the finite meets of elements of $M^\infty(A)$ is denoted by $M^\infty_\omega(A)$ (so that 0 belongs to $J^\infty_\omega(A)$ but does not belong to $J^\infty(A)$ and 1 belongs to $M^\infty_\omega(A)$ but does not belong to $M^\infty(A)$).

In the variety of bounded distributive lattices, the class of doubly algebraic lattices can be characterized in different ways.

LEMMA 4.2. *Assume that A is a complete $\mathcal{DL}$. Then, the following conditions are equivalent.*

(1) *A is doubly algebraic,*
(2) *A is algebraic and every element of A is a join of elements of $J^\infty(A)$,*
(3) *A is completely distributive and every element of A is a join of elements of $J^\infty(A)$,*
(4) *A is isomorphic to the lattice of the isotone maps from $J^\infty(A)$ to the two element chain.*
(5) *there is a poset P such that A is isomorphic to the lattice of isotone maps from P to the two element chain.*

The canonical extension of a $\mathcal{DL}$ can be described in two different ways. We use the following as a definition.

DEFINITION 4.3. The *canonical extension A^σ of a $\mathcal{DL}$ A* is defined, up to isomorphism, as the lattice of isotone maps from the PRIESTLEY dual of A to the two element chain.

Apparently, this vocabulary and notation collide with the previously introduced canonical extension of an $\mathcal{MMV}^{\mathcal{L}}_n$-algebra. The frightened reader may have a glimpse at Proposition 4.26 that makes this not so schizophrenic use of the vocabulary possible.

Thanks to Lemma 4.2, we get the following result easily.

LEMMA 4.4. *The canonical extension of a $\mathcal{DL}$ A is a bounded doubly algebraic lattice.*

DEFINITION 4.5. We denote by $\mathcal{DL}^+$ the class of doubly algebraic bounded distributive lattices.

Even if the proposed definition is concrete, it turns out that is it more convenient to characterized the canonical extension A^σ of a distributive lattice A by properties involving A and A^σ. This characterization requires the following definitions.

DEFINITION 4.6. A sublattice A of a $\mathcal{DL}^+$ B is *a separating sublattice of* B if for any p in $J^\infty(B)$ and u in $M^\infty(B)$ such that $p \leq u$, the interval $[p, u]$ contains an element of A.

The sublattice A is *compact* in B if for any subset S and T of A such that $\bigwedge S \leq \bigvee T$, there are a finite subset S' of S and a finite subset T' of T such that $\bigwedge S' \leq \bigvee T'$.

The following result is a useful criterion to recognize the canonical extension of a bounded distributive lattice.

PROPOSITION 4.7. *If A is a $\mathcal{DL}$, then A is a compact separating sublattice of its canonical extension A^σ. Moreover if B is a $\mathcal{DL}^+$ that contains A as a separating compact sublattice, then there is a unique isomorphism f from A^σ to B such that $f\restriction_A = \mathrm{id}_A$.*

As a consequence, we obtain for example the following lemma. We denote by A^α the *order dual* of A for any poset A.

LEMMA 4.8. *If $A_1, \ldots, A_n$ are $\mathcal{DL}$s then*

(1) $({A_1}^\alpha)^\sigma$ *is equal to* $(A_1^\sigma)^\alpha$,
(2) $(A_1 \times \cdots \times A_n)^\sigma$ *is equal to* $A_1^\sigma \times \cdots \times A_n^\sigma$.

DEFINITION 4.9. Assume that A is a $\mathcal{DL}$ that is a sublattice of a $\mathcal{DL}^+$ B. A *closed element* of B is an element that can be obtained as a meet of elements of A. An *open element* of B is an element that can be obtained as a join of elements of A. We denote by $K(B)$ the set of the closed elements of B and by $O(B)$ the set of the open elements of B.

The preceding definition makes reference to a subalgebra A of B. But for the sake of readability, we decide not to recall that dependence in the notations. Actually, in the sequel, we only use that definition for a $\mathcal{DL}$ A with $B = A^\sigma$. In that case, since A is a separating subalgebra of A^σ, we obtain that $J_\omega^\infty(A^\sigma) \subseteq K(A^\sigma)$ and $M_\omega^\infty(A^\sigma) \subseteq O(A^\sigma)$.

In order to define extension of maps and to study preservations of identities through canonical extension, we need to add topologies to the canonical extension of a $\mathcal{DL}$ A. The first family of topologies is defined without any reference to A, *i.e.*, their definition involves only the fact that A^σ is a doubly algebraic lattice.

DEFINITION 4.10. If B is a $\mathcal{DL}^+$, then the topologies $\iota^\uparrow$, $\iota^\downarrow$ and ι are defined as the topologies that have for base the sets $[p)$, $(u]$ and $[p) \cap (u]$ respectively where p ranges in $J_\omega^\infty(B)$ and u ranges in $M_\omega^\infty(B)$.

The second family of topologies involves A in their definition.

DEFINITION 4.11. If A is a $\mathcal{DL}$, then the topologies $\sigma^\uparrow$, $\sigma^\downarrow$ and σ are defined on A^σ as the topologies that have respectively for base the sets $[p)$, $(u]$ and $[p) \cap (u]$ where p ranges in $K(A^\sigma)$ and $u \in O(A^\sigma)$.

We obtain directly that $\iota^\uparrow \subseteq \sigma^\uparrow$, $\iota^\downarrow \subseteq \sigma^\downarrow$ and $\iota \subseteq \sigma$. A continuous map $f : \langle A^\sigma, t\rangle \to \langle B^\sigma, s\rangle$ where s and t are among theses topologies is called a *(s,t)-continuous map.* Let us note the following important result about σ.

LEMMA 4.12. *The topological structure $(A^\sigma, \leq, \sigma)$ is a totally order disconnected space and A is a dense subspace of A^σ. The elements of A are exactly the isolated points of A^σ.*

1.1. Canonical extensions of $\mathcal{DL}$ maps. Since we are interested in *expanded* $\mathcal{DL}$s (such as for example MV_n-algebras with dual $\mathcal{L}$-operators), it is important to define a canonical way to extend a map between two $\mathcal{DL}$s A and B into a map between A^σ and B^σ. We use the density of A in A^σ to define such an extension as a *limit superior* or a *limit inferior.* Recall the following definition.

DEFINITION 4.13. If (X, τ) is a topological space, if Y is a dense subset of X and if B is a $\mathcal{DL}^+$, then for any map $f : Y \to B$, the map $\liminf_\tau f$ is defined by

$$\liminf_\tau f : X \to C : x \mapsto \bigvee\{\bigwedge f(U \cap Y) \mid x \in U \in \tau\}$$

and $\limsup_\tau f$ by

$$\limsup_\tau f : X \to C : x \mapsto \bigwedge\{\bigvee f(U \cap Y) \mid x \in U \in \tau\}.$$

We can now define two extensions of a map $f : A \to B$ between two $\mathcal{DL}$s A and B to a map between A^σ and B^σ.

DEFINITION 4.14. If $f : A \to B$ is a map between two $\mathcal{DL}$s A and B then the maps $f^\sigma : A^\sigma \to B^\sigma$ and $f^\pi : A^\sigma \to B^\sigma$ are defined by

$$f^\sigma = \liminf_\sigma f \quad \text{and} \quad f^\pi = \limsup_\sigma f.$$

and are respectively called *the lower (canonical) extension of f* and the *upper (canonical) extension of f*.

It may be useful to note that for any $f : A \to B$, the extensions f^σ and f^π can be computed in that way:

$$f^\sigma(x) = \bigvee\{\bigwedge\{f(a) \mid p \leq a \leq u\} \mid p \leq x \leq u, p \in K(A^\sigma) \text{ and } u \in O(A^\sigma)\},$$

and

$$f^\pi(x) = \bigwedge\{\bigvee\{f(a) \mid p \leq a \leq u\} \mid p \leq x \leq u, p \in K(A^\sigma) \text{ and } u \in O(A^\sigma)\},$$

for any x in A^σ. Moreover, if f is an isotone map,

$$f^\sigma(x) = \bigvee\{\bigwedge\{f(a) \mid p \leq a \in A\} \mid x \geq p \in K(A^\sigma)\}$$

and

$$f^\pi(x) = \bigwedge\{\bigvee\{f(a) \mid u \geq a \in A\} \mid x \leq u \in O(A^\sigma)\}.$$

The following lemma proves that the maps f^σ and f^π are extensions of f.

LEMMA 4.15. *If $f : A \to B$ is a map between two $\mathcal{DL}$s A and B then*

(1) *the maps f^σ and f^π are extensions of f,* i.e., $f^\sigma\lfloor_A = f^\pi\lfloor_A = f$,
(2) *the map f^σ is the largest $(\sigma, i^\uparrow)$-continuous extension of f to A^σ and f^π is the smallest $(\sigma, \iota^\downarrow)$-continuous extension of f to A^σ.*

The canonical extension of an expanded bounded distributive lattice A will be defined as the canonical extension of the $\mathcal{DL}$-reduct of A equipped with a canonical extension of the non-lattice operations. The ideal case arises when we do not have to choose between the upper and the lower extensions.

DEFINITION 4.16. A map $f : A \to B$ between two $\mathcal{DL}$s A and B is *smooth* if $f^\sigma = f^\pi$.

For example, it is proved in Proposition 4.26 that the maps $\oplus$ and $\neg$ are smooth in the variety of MV_n-algebras.

LEMMA 4.17. *A map $f : A \to B$ between two $\mathcal{DL}$s A and B is smooth if and only if f^σ is (σ, ι)-continuous. Conversely, any map $f : A \to B$ that admits a (σ, ι)-continuous extension $g : A^\sigma \to B^\sigma$ is smooth and $f^\sigma = g$.*

Thanks to the definition of the extensions of maps between $\mathcal{DL}$s, we can define the canonical extensions of a bounded distributive expansion.

DEFINITION 4.18. If $A = \langle A, \{f_i \mid i \in I\}\rangle$ is a bounded distributive expansion of the $\mathcal{DL}$ A, then the *canonical extension* A^σ of A is the algebra $\langle A^\sigma, \{f_i^\sigma \mid i \in I\}\rangle$ and the *dual canonical extension* A^π of A is the algebra $\langle A^\sigma, \{f_i^\pi \mid i \in I\}\rangle$.

Note that if A^α denotes the algebra $\langle A^\alpha, \{f_i \mid i \in I\}\rangle$, then $A^\pi = A^{\alpha\sigma\alpha}$.

A map is join preserving if it preserves all binary (and thus all finite non empty) joins. A map is completely join preserving if it preserves all non empty join. So, we do not require that such maps preserve 0 and 1. But, we require that (complete) $\mathcal{DL}$-homomorphisms do.

DEFINITION 4.19. A map $f : A_1 \times \cdots \times A_n \to B$ between $\mathcal{DL}$s $A_1, \ldots, A_n, B$ is a *lattice operator* or simply an *operator* if f is join preserving in each of its coordinate. Similarily, the map f is a *complete lattice operator* or simply a *complete operator* if it is completely join preserving in each of its coordinate. The map f is a *dual (complete) operator* if $f : A_1^\alpha \times \cdots \times A_n^\alpha \to B^\alpha$ is a (complete) operator.

We denote by $\mathcal{MVO}_n^{\mathcal{L}}$ the variety of *MV_n-algebras with $\mathcal{L}$-lattice operators*, *i.e.*, the variety of algebras over the language $\mathcal{L}$ whose MV-reduct belongs to MV_n and such that any f in $\mathcal{L} \setminus \{\oplus, \odot, \neg\}$ is interpreted as a dual lattice operator on A.

The main result about canonical extension of operators is the following one.

PROPOSITION 4.20. *The canonical extension of an operator is a complete operator.*

In the approach of preservations of identities through canonical extensions, the following results about continuity of the canonical extensions of maps are widely applied.

PROPOSITION 4.21. *If $f : A \to B$ is a map between two $\mathcal{DL}$s A and B,*

(1) *if f is isotone then f^σ is isotone and $(\sigma^\uparrow, \iota^\uparrow)$-continuous,*
(2) *if f is an operator then the map f^σ is $(\iota^\uparrow, \iota^\uparrow)$-continuous,*
(3) *if f is join preserving then f^σ is completely join preserving and is $(\sigma^\downarrow, \sigma^\downarrow)$-continuous*
(4) *if f is meet preserving and join preserving then f^σ is (σ, σ)-continuous.*

Let us also state the following result about composition of extensions.

PROPOSITION 4.22. *If $f : B \to C$ and $g : A \to B$ are two maps between $\mathcal{DL}$s A, B and C,*

(1) *if f and g are isotone maps then $(fg)^\sigma \leq f^\sigma g^\sigma$,*
(2) *if $f^\sigma g^\sigma$ is $(\sigma, \iota^\uparrow)$-continuous then $(fg)^\sigma \geq f^\sigma g^\sigma$,*
(3) *if $f^\sigma g^\sigma$ is $(\sigma, \iota^\downarrow)$-continuous then $(fg)^\sigma \leq f^\sigma g^\sigma$,*
(4) *if f is join preserving and meet preserving then $(fg)^\sigma = f^\sigma g^\sigma$,*
(5) *if g is join preserving and meet preserving then $f^\sigma g^\sigma \leq (fg)^\sigma$,*
(6) *if g is join preserving, meet preserving and onto then $(fg)^\sigma = f^\sigma g^\sigma$.*

Assume that A is a bounded lattice expansion. The set of the terms t whose term function t^A on A satisfies $(t^A)^\sigma = t^{A^\sigma}$ is of particular interest. Indeed if t and s are two such terms and if if $t^A = s^A$ it follows that $t^{A^\sigma} = (t^A)^\sigma = (s^A)^\sigma = s^{A^\sigma}$. Thus, the equation $s = t$ is also satisfied in A^σ. This piece of argument justifies the following definition.

DEFINITION 4.23. Assume that $\mathcal{L}$ is an expansion of the language $\{\vee, \wedge, 0, 1\}$ of bounded distributive lattices. We denote by $\mathcal{DLE}_{\mathcal{L}}$ the *variety of the distributive lattice $\mathcal{L}$-expansions*, *i.e.*, the variety of the algebras over the language $\mathcal{L}$ whose reduct to $\{\vee, \wedge, 0, 1\}$ is a $\mathcal{DL}$.

If A belongs to $\mathcal{DLE}_{\mathcal{L}}$, an $\mathcal{L}$-term t is *expanding* on A if $(t^A)^\sigma \leq t^{A^\sigma}$. It is *contracting* on A if $(t^A)^\sigma \geq t^{A^\sigma}$ and *stable* if $(t^A)^\sigma = t^{A^\sigma}$.

A variety $\mathcal{A}$ of $\mathcal{DLE}_{\mathcal{L}}$s is *canonical* if it contains the canonical extension of its members and if the canonical extension of an $\mathcal{L}$-homomorphism between two algebras of A is an $\mathcal{L}$-homomorphism.

When we are not in the context of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras, we reserve the notation $\mathcal{L}$ to denote an expansion of the language $\{\vee, \wedge, 0, 1\}$ if no other specification is given. The following results will turn out to be very useful.

PROPOSITION 4.24. *Assume that A is a $\mathcal{DLE}_{\mathcal{L}}$ and that t is an $\mathcal{L}$-term. If for any operation symbol f that occurs in t, the map f^A is isotone, then t is expanding on A.*

PROOF. The proof is an easy induction on the number of connectives in t with the help of the first item of Proposition 4.22. □

1.2. Canonical extensions of MV$_n$-algebras with operators. It is time to apply the previous results to the algebras of this dissertation.

The first result is about canonical extensions of MV$_n$-algebras. It can be obtained as a direct consequence of Theorem 3.15 in [**20**], but we provide a stand-alone proof. Note that $Ł_n^\sigma = Ł_n$ for any positive integer n. In the sequel, if $f : A \to B$ is a map, we denote by $f^{[k]}$ the map $f^{[k]} : A^k \to B^k : (a_1, \ldots, a_k) \mapsto (f(a_1), \ldots, f(a_k))$.

LEMMA 4.25. *Assume that A is an MV$_n$-algebra. The canonical extension of any element of $\mathcal{MV}(A, Ł_n)$ is an element of $\mathcal{MV}(A^\sigma, Ł_n)$.*

PROOF. Let us denote by m a positive divisor of n such that $u : A \twoheadrightarrow Ł_m$. We prove that $u^\sigma : A^\sigma \to Ł_m$ is an homomorphism. Indeed, if f belongs to $\{\oplus^A, \neg^A\}$ is of arity k, we obtain successively

$$u^\sigma f^\sigma = (uf)^\sigma = (fu^{[k]})^\sigma = f^\sigma u^{[k]\sigma} = f^\sigma u^{\sigma[k]}$$

where the first equality is obtained thanks to item (4) of Proposition 4.22, the third thanks to its item (6) and where the other equalities are trivial. □

PROPOSITION 4.26. *If A is an MV$_n$-algebra, then the canonical extension A^σ of A is*

$$\prod_{u \in \mathcal{MV}(A, Ł_n)} u(A).$$

The operations $\oplus^A$ and $\neg^A$ are smooth and the variety of MV$_n$-algebras is canonical.

PROOF. The underlying lattice of the MV$_n$-algebra

$$A' = \prod_{u \in \mathcal{MV}(A, Ł_n)} u(A)$$

is clearly a doubly algebraic lattice. Recall that $J^\infty(A')$ contains exactly the elements x of A' for which there is a u in $\mathcal{MV}(A, Ł_n)$ such that $x_u \neq 0$ and $x_v = 0$ if $v \neq u$ belongs to

$\mathcal{MV}(A, Ł_n)$. Similarly $M^\infty(A')$ contains exactly the elements y of A' for which there is a u in $\mathcal{MV}(A, Ł_n)$ such that $y_u \neq 1$ and $y_v = 1$ if $v \neq u$ belongs to $\mathcal{MV}(A, Ł_n)$.

Assume then that x belongs to $J^\infty(A')$, that y belongs to $M^\infty(A')$, that $x \leq y$ and that u, u' are elements of $\mathcal{MV}(A, Ł_n)$ such that $x_u \neq 0$ and $y_{u'} \neq 1$. If $u \neq u'$, since

$$e_A : A \hookrightarrow A' : a \mapsto (u(a))_{u \in \mathcal{MV}(A,Ł_n)}$$

is a boolean representation of A, there are two elements a and b in A such that $a_u = x_u$ and $b_{u'} = y_{u'}$. If Ω is a clopen subset of $\mathcal{MV}(A, Ł_n)$ that contains u but does not contain u', then

$$a_\Omega \cup b_{\mathcal{MV}(A,Ł_n)\setminus\Omega}$$

is an element of A that belongs to $[x, y]$. If $u = u'$, there is an element a of A such that $a_u = x_u$ and so that belongs to $[x, y]$. We thus have proved that A is a separating sublattice of A'.

Let us now prove that A is compact in A'. Let us assume that X and Y are two subsets of A with $\bigwedge X \leq \bigvee Y$. Then, for any $u \in \mathcal{MV}(A, Ł_n)$, there is an $x^{(u)}$ is X and a $y^{(u)}$ in Y such that $(x^{(u)})_u \leq (y^{(u)})_u$. Then, the sets

$$\Gamma_u = \bigcup_{\frac{i}{n}=(x^{(u)})_u}^{1} [x^{(u)} : \frac{i}{n}] \cup \bigcup_{\frac{i}{n}=0}^{(y^{(u)})_u} [y^{(u)} : \frac{i}{n}]$$

form an open covering of $\mathcal{MV}(A, Ł_n)$ if u ranges through $\mathcal{MV}(A, Ł_n)$. Thus, there are some $u_1, \ldots, u_k$ in $\mathcal{MV}(A, Ł_n)$ such that $\{\Gamma_{u_1}, \ldots, \Gamma_{u_k}\}$ is still a covering of $\mathcal{MV}(A, Ł_n)$. It follows that, for any v in $\mathcal{MV}(A, Ł_n)$, there is a i_0 in $\{1, \ldots, k\}$ such that $v \in \Gamma_{i_0}$ and such that

$$\bigwedge_{1 \leq i \leq k} (x^{(u_i)})_v \leq (x^{(u_{i_0})})_{u_{i_0}} \leq (y^{(u_{i_0})})_{u_{i_0}} \leq \bigvee_{1 \leq i \leq k} (y^{(u_i)})_v$$

and that proves that A is compact in A'.

We now prove that $\oplus^{A'}$ coincides with $\oplus^\sigma$. It follows successively that for any x and y in A',

$$\begin{aligned} x \oplus^\sigma y &= \bigvee\{\bigwedge\{a \oplus^{A'} a' \mid a \in [p) \cap A, a' \in [p') \cap A\} \mid p, p' \in K(A'), p \leq x, p' \leq y\} \\ &= \bigvee\{(\bigwedge_{a \in [p) \cap A} a) \oplus^{A'} (\bigwedge_{a' \in [p) \cap A} a') \mid p, p' \in K(A'), p \leq x, p' \leq y\} \\ &= \bigvee\{p \oplus^{A'} p' \mid p, p' \in K(A'), p \leq x, p' \leq y\} \\ &= x \oplus^{A'} y, \end{aligned}$$

thanks to the isotony of $\oplus^A$ and the fact that of A' is a complete and completely distributive MV-algebra. We proceed in a similar way to conclude that $\neg^\sigma = \neg^{A'}$.

Let us proof that $\oplus^A$ is smooth. It follows successively that

$$\begin{aligned} (\oplus^A)^\pi &= \limsup_{\sigma((A^\sigma)^2)} \oplus^A = \liminf_{\sigma((A^{\sigma\alpha})^2)} \oplus^A \\ &= \liminf_{\sigma((A^{\alpha\sigma})^2)} \oplus^A = \liminf_{\sigma(A^{\alpha\sigma})^2)} \odot^{A^\alpha} \\ &= \odot^{A^{\alpha\sigma}}. \end{aligned}$$

Then, $\odot^{A^{\alpha\sigma}}$ coincides with the operation $\odot$ on

$$\prod_{u \in \mathcal{MV}(A^{\alpha}, \L_n)} u(A^{\alpha})$$

and is equal to the operation $\oplus$ on

$$\prod_{u \in \mathcal{MV}(A, \L_n)} u(A).$$

Thus, the operation $\odot^{A^{\alpha\sigma}}$ coincides with $(\oplus^{A})^{\sigma}$.

Let us now prove that if $h : A \to B$ is an homomorphism between two MV_n-algebras A and B then $h^{\sigma} : A^{\sigma} \to B^{\sigma}$ is an homomorphism. For any u in $\mathcal{MV}(B, \L_n)$, the map $p_u \restriction_B \circ h : A \to u(B)$ is an homomorphism. Thanks to Lemma 4.25, we obtain that $(p_u \restriction_B \circ h)^{\sigma} : A \to u(B)$ is an homomorphism. Thus, by item (4) of Proposition 4.22, the map $p_u \restriction_B^{\sigma} \circ h^{\sigma}$ is an homomorphism. It follows that the map

$$h' : A^{\sigma} \to B^{\sigma} : x \mapsto (p_u \restriction_B^{\sigma} (h^{\sigma}(x))_{u \in \mathcal{MV}(B, \L_n)}$$

is an homomorphism. The conclusion follows from the fact that $h' = h^{\sigma}$ since $p_u \restriction_B^{\sigma} = p_u$. $\square$

Definition 4.27. If $\langle X, \tau \rangle$ is a topological space and if Y is a subset of X, we denote by $\tau \restriction_Y$ the topology induced by X on Y.

Corollary 4.28. *If A is an MV_n-algebra, there is a unique isomorphism $\phi : \mathfrak{B}(A)^{\sigma} \to \mathfrak{B}(A^{\sigma})$ with $\phi(a) = a$ for any a in $\mathfrak{B}(A)$. Morevover, this map ϕ is an homeomorphism between $\langle \mathfrak{B}(A)^{\sigma}, s(\mathfrak{B}(A)^{\sigma}) \rangle$ and $\langle \mathfrak{B}(A^{\sigma}), s(A^{\sigma}) \restriction_{\mathfrak{B}(A^{\sigma})} \rangle$ for any s in $\{\iota^{\uparrow}, \iota^{\downarrow}, \iota, \sigma^{\uparrow}, \sigma^{\downarrow}, \sigma\}$.*

Proof. We may for example obtain the isomorphism ϕ thanks to Proposition 4.26 and the unicity of ϕ follows from Proposition 4.7 . Clearly, this isomorphism sends closed, open, completely meet irreducible and completely join irreducible elements to closed, open, completely meet irreducible and completely join irreducible elements respectively and conversely.

Then, if p belongs to $K(A^{\sigma})$, it follows that $\mathfrak{B}(A^{\sigma}) \cap [p) = \mathfrak{B}(A^{\sigma}) \cap [n.p)$ and $\phi^{-1}(\mathfrak{B}(A^{\sigma}) \cap [p)) = [\phi^{-1}(n.p))$. Since $n.p = n. \bigwedge\{a \mid p \leq a \in A\} = \bigwedge\{n.a \mid p \leq a \in A\}$ is a closed element of A^{σ}, we have proved that $\phi : \langle \mathfrak{B}(A)^{\sigma}, \sigma^{\uparrow}(\mathfrak{B}(A)^{\sigma}) \rangle \to \langle \mathfrak{B}(A^{\sigma}), \sigma^{\uparrow}(A^{\sigma}) \restriction_{\mathfrak{B}(A^{\sigma})} \rangle$ is continuous.

Now, if p belongs to $K(\mathfrak{B}(A)^{\sigma})$, then $\phi(p^{\uparrow}) = \phi(p)^{\uparrow}$ which proves that ϕ^{-1} is continuous and so that ϕ is an homeomorphism.

We proceed in a similar way for the other topologies. $\square$

Here is a sample illustration of the connections that exists between A^{σ} and $\mathfrak{B}(A)^{\sigma}$.

Proposition 4.29. *Assume that A is an MV_n-algebra and that i is in $\{1, \ldots, n\}$. If p belongs to $K(A^{\sigma})$ then $\tau_{i/n}(p)$ belongs to $K(\mathfrak{B}(A)^{\sigma})$.*

Proof. The map $(\tau_{i/n}^{A})^{\sigma}$ is completely meet-preserving since $\tau_{i/n}^{A}$ is meet preserving. Thanks to a repeated application of item (4) of Proposition 4.22, we obtain that $(\tau_{i/n}^{A})^{\sigma} = (\tau_{i/n})^{A^{\sigma}}$. Thus, for any element p in A^{σ} that satisfies $p = \bigwedge\{a \in A \mid p \leq a\}$, we have

$$\tau_{i/n}(p) = \bigwedge\{\tau_{i/n}(a) \mid p \leq a \in A\}.$$

Hence, the element $\tau_{i/n}(p)$ can be written as a complete meet of elements of $\mathfrak{B}(A^{\sigma}) = \mathfrak{B}(A)^{\sigma}$ and the conclusion follows. $\square$

From now on, we will often prove results with the help of the representation of A^σ that is proposed in Proposition 4.26. For example, we obtain easily that if p belongs to $J^\infty_\omega(A^\sigma)$ then $\tau_{i/n}(p)$ belongs to $J^\infty_\omega(\mathfrak{B}(A^\sigma)) = J^\infty_\omega(\mathfrak{B}(A)^\sigma)$.

DEFINITION 4.30. If $A = \langle A, \{f_i \mid i \in I\}\rangle$ is an MV_n-algebra with dual $\mathcal{L}$-operators, the *canonical extension of* A is the algebra $\langle A^\sigma, \{f_i^\sigma \mid i \in I\}\rangle$.

Apparently, this freshly introduced vocabulary collides with existing one. We indeed have already introduced the canonical extension of an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra A as the algebra $(A_{\times_n})^{\times_n}$ in Definition 2.44. It is imperative to prove that Definition 2.44 and Definition 4.30 coincide.

PROPOSITION 4.31. *If A is an MV_n-algebra with $\mathcal{L}$-operators, then for any k-ary dual MV-operator ∇ of $\mathcal{L}$ with canonical relation R, any $\alpha_1, \ldots, \alpha_k$ in A^σ and any u in $\mathcal{MV}(A, \text{Ł}_n)$,*

$$(\nabla^\sigma(\alpha_1, \ldots, \alpha_n))_u = \bigwedge_{(u,v_1,\ldots,v_k)\in R}\ \bigvee_{1\le i\le k} \alpha_i(v_i).$$

Consequently, the canonical extension A^σ of A is isomorphic to the algebra $(A_{\times_n})^{\times_n}$. Moreover, the variety $\mathcal{MMV}_n^{\mathcal{L}}$ is canonical.

PROOF. Let us denote by ∇_R the operation defined on A^σ by

$$(\nabla_R(\alpha_1, \ldots, \alpha_n))_u = \bigwedge_{(u,v_1,\ldots,v_k)\in R}\ \bigvee_{1\le i\le k} \alpha_i(v_i),$$

for any $\alpha_1, \ldots, \alpha_k$ in A^σ and any u in $\mathcal{MV}(A, \text{Ł}_n)$, *i.e.*, the operation ∇_R is the operation defined in Definition 2.33. We already know that ∇_R and ∇^σ are extensions of ∇^A. Now, if $p_1, \ldots, p_k$ are closed elements of A^σ, it follows that

$$\begin{aligned}\nabla^\sigma(p_1, \ldots, p_k) &= \bigwedge\{\nabla(a_1, \ldots, a_k) \mid (p_1, \ldots, p_k) \le (a_1, \ldots, a_k) \in A^\sigma\}\\ &= \bigwedge\{\nabla_R(a_1, \ldots, a_k) \mid (p_1, \ldots, p_k) \le (a_1, \ldots, a_k) \in A^\sigma\}\\ &= \nabla_R(p_1, \ldots, p_k),\end{aligned}$$

since ∇_R is completely meet preserving on each of its arguments. Then, if $\alpha_1, \ldots, \alpha_k$ are elements of A^σ,

$$\begin{aligned}\nabla^\sigma(\alpha_1, \ldots, \alpha_k) &= \bigvee\{\nabla^\sigma(p_1, \ldots, p_k) \mid (\alpha_1, \ldots, \alpha_k) \ge (p_1, \ldots, p_k) \in K(A^\sigma)\}\\ &= \bigvee\{\nabla_R(p_1, \ldots, p_k) \mid (\alpha_1, \ldots, \alpha_k) \ge (p_1, \ldots, p_k) \in K(A^\sigma)\}.\end{aligned}$$

If u belongs to $\mathcal{MV}(A, \text{Ł}_n)$, we obtain that $(\nabla^\sigma(\alpha_1, \ldots, \alpha_k))_u$ is equal to

$$\bigvee\{(\nabla_R(p_1, \ldots, p_k))_u \mid (\alpha_1, \ldots, \alpha_k) \ge (p_1, \ldots, p_k) \in K(A^\sigma)\},$$

so to

$$\bigvee\{\bigwedge\{(p_1)_{v_1} \vee \cdots \vee (p_k)_{v_k} \mid (u, v_1, \ldots, v_k) \in R\} \mid (\alpha_1, \ldots, \alpha_k) \ge (p_1, \ldots, p_k) \in K(A^\sigma)\}$$

and to

$$\bigwedge\{\bigvee\{(p_1)_{v_1} \vee \cdots \vee (p_k)_{v_k} \mid (\alpha_1, \ldots, \alpha_k) \ge (p_1, \ldots, p_k) \in K(A^\sigma)\} \mid (u, v_1, \ldots, v_k) \in R\}.$$

This last element is by definition equal to

$$\bigwedge\{(\alpha_1)_{v_1} \vee \cdots \vee (\alpha_k)_{v_k} \mid (u, v_1, \ldots, v_k) \in R\}$$

and eventually to $(\nabla_R(\alpha_1, \ldots, \alpha_k))_u$.

Let us now assume that $h : A \to B$ is an $\mathcal{L}$-homomorphism. We have to prove that $h^\sigma : A^\sigma \to B^\sigma$ is an $\mathcal{L}$-homomorphism. According to Proposition 4.26, we just have to prove that $h^\sigma \nabla^\sigma = \nabla^\sigma h^{\sigma[k]}$ for any k-ary dual MV-operator ∇ of $\mathcal{L}$. This result is obtained thanks to the sequence of identities

$$h^\sigma \nabla^\sigma = (h\nabla)^\sigma = (\nabla h^{[k]})^\sigma = \nabla^\sigma h^{[k]\sigma} = \nabla^\sigma h^{\sigma[k]}$$

in which the second and the last identities are trivial, the first one is obtained by item (4) of Proposition 4.22 and the third one by item (1) and item (5) of the same result. □

We aslo have to prove that Definition 3.42 and Definition 4.23 of a canonical variety of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras coincide.

COROLLARY 4.32. *A variety $\mathcal{A}$ of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras is canonical in the sense of Definition 3.42 if and only if $\mathcal{A}$ is in the sense of Definition 4.23 of any of its member.*

2. Back to canonicity

We have already emphasized the importance of canonicity in the generation of Ł$_n$-valued KRIPKE complete logics. Thanks to the previous developments, we are now able to provide a proof of the following result, which was announced in subsection 4.4.

PROPOSITION 4.33. *An Ł$_n$-valued modal logic **L** is canonical if and only if the variety of **L**-algebras is canonical.*

PROOF. The proof is now a routine argument now that we know that quotient maps are preserved through canonical extensions. □

2.1. SAHLQVIST formulas to defined canonical varieties. We have reduced the problem of finding canonical Ł$_n$-valued modal $\mathcal{L}$-logics to the problem of finding canonical varieties of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras. A classical way to produce such varieties is to study the preservation of equations through canonical extensions. Indeed, any set of equations that is preserved under canonical extension defines a canonical variety.

SAHLQVIST formulas are a family of formulas over the language of boolean algebras with operators that are preserved under canonical extensions. They were introduced by SAHLQVIST in [**50**]. The algebraic treatment of this family of formulas was considered in [**32**]. This success lead mathematicians to consider so called "SAHLQVIST formulas" in wider contexts (*e.g.*, [**21, 23, 12**]).

We here adapt the classical results about SAHLQVIST formulas and normal modal logics to Ł$_n$-valued modal normal logics. The algebraic approach makes this adaptation quite painless.

For our purposes, it is important to set of primitive operations that we consider to define algebras. So, we are going to denote by $\mathcal{L}_{MMV}$ a set $\{\odot, \vee, \neg, 0, 1\} \cup \{f_i \mid i \in I\}$ where $\odot$, $\vee$ are binary, the negation $\neg$ is unary and f_i is of arity k_i for any i in I.

The language $\mathcal{L}^d_{MMV}$ is the language $\mathcal{L}_{MMV} \cup \{\oplus, \wedge\} \cup \{f_i^d \mid i \in I\}$, where $\oplus$ and $\wedge$ are binary and f_i^d is of arity k_i for any $i \in I$.

The intended meaning of the operations $\oplus$, $\odot$, $\neg$, 0, 1 is clear. These are going to be interpreted as the MV-algebra operations. Unless stated otherwise, we do not require any special property on the operations f_i with i in I. But, when we deal with algebras and terms of the language $\mathcal{L}_{MMV^d}$, we restrict ourself to algebras that satisfy the following equations

$$x \wedge y = \neg(\neg x \vee \neg y), \quad x \oplus y = \neg(\neg x \odot \neg y), \tag{2.1}$$

and

$$f_i^d(x_1, \ldots, x_{k_i}) = \neg f_i(\neg x_1, \ldots, \neg x_{k_i}) \tag{2.2}$$

for any i in I. More generally, if $g : B_1 \times \cdots \times B_k \to A$ is a map (a term function for example), then we denote by g^d the map

$$g^d : B_1 \times \cdots \times B_k \to A : (x_1, \ldots, x_k) \mapsto \neg g(\neg x_1, \ldots, \neg x_k),$$

which is called the *dual map of g*, or simply the *dual of g*. The key idea is that by applying equations (2.1) and (2.2) to an $\mathcal{L}_{MMV}$-term τ, we are able to produce an equivalent $\mathcal{L}_{MMV^d}$-term τ' that contains a considerably smaller number of negation symbols. This idea is made clear in the sequel.

The following vocabulary was introduced in [**32, 23**].

Definition 4.34. Let $\mathcal{L}$ be the language $\mathcal{L}_{MMV}$ or $\mathcal{L}_{MMV^d}$. An $\mathcal{L}$-term τ is

- *positive primitive* if it is a constant term (*i.e.*, without variable) or if it is equal to $f(x_1, \ldots, x_k)$ for an k-ary operation f of $\mathcal{L} \setminus \{\neg\}$;
- *strictly positive* if no variable of τ is in the scope of any negation symbol (thus, the negation symbols have constant terms as arguments);
- *positive* if every variable of τ is in the scope of an even number of negation symbol;
- *negative* if every variable of τ is in the scope of an odd number of negation symbols.

We denote by τ^* the term obtained from τ by switching every operation that appears in τ by its dual operation.

If $\mathcal{A}$ is a class of $\mathcal{L}$-algebras, two terms τ and τ' are said *$\mathcal{A}$-equivalent* (or simply *equivalent* if $\mathcal{A}$ is the variety of $\mathcal{L}$-algebras) if the term functions τ^A and τ'^A are equal on every algebra A of $\mathcal{A}$ (that satisfies, following our convention, equations (2.1) and (2.2) if $\mathcal{L} = \mathcal{L}_{MMV^d}$).

Note that it is possible to give inductive definitions of the preceding classes of terms. Such definitions would provide a good support for the proofs. But since the proofs of the results that we are going to use can be found in [**23**], we do not bother with such definitions, neither with the proofs. These results are anyway easy to accept without any proof.

Lemma 4.35. *If τ is a term over $\mathcal{L}_{MMV}$ or $\mathcal{L}_{MMV_d}$ then*

1. *the term τ is equivalent to a positive (resp. negative) term if and only if τ^d is equivalent to a positive (resp. negative) term.*
2. *If $\sigma_1, \ldots, \sigma_n$ are terms then $(\tau(\sigma_1, \ldots, \sigma_n))^d = \tau^d(\sigma_1^d, \ldots, \sigma_n^d)$.*
3. *If τ is an $\mathcal{L}_{MMV^d}$-term then it is equivalent to a $\mathcal{L}_{MMV^d}$-term written in* standard form, *that is an $\mathcal{L}_{MMV^d}$ term in which the negation symbols appear next to constant terms or directly next to variables.*

The last item of the preceding lemma is already a useful simplification of $\mathcal{L}_{MMV^d}$-terms.

DEFINITION 4.36. Let us denote by Ψ_0 the smallest set of $\mathcal{L}_{MMV}$-terms that contains the positive primitive terms and that is closed under substitution, and by Ψ the smallest set of $\mathcal{L}_{MMV^d}$-terms that contains the positive primitive terms *and their dual terms* and which is closed under substitution.

PROPOSITION 4.37. *With the previous definitions in mind,*

(1) *an $\mathcal{L}_{MMV}$-term is equivalent to a strictly positive $\mathcal{L}_{MMV}$-term if and only if it is equivalent to a term of Ψ_0,*
(2) *an $\mathcal{L}_{MMV}$-term is equivalent to a positive $\mathcal{L}_{MMV}$-term if and only if it is equivalent to a term of Ψ,*
(3) *an $\mathcal{L}_{MMV}$-term is equivalent to a negative $\mathcal{L}_{MMV}$-term if and only if it is equivalent to the negation of a term of Ψ.*

The preceding Proposition allows us to restrict ourself to the terms of Ψ when we deal with positive terms.

Let us define the class of SAHLQVIST equations for the $Ł_n$-valued modal logics.

DEFINITION 4.38. A *box* is a unary dual MV-operator. A *boxed atom* is a variable preceded by a string of boxes.

A SAHLQVIST *equation* is an equation $\phi \leq \psi$ where

- ψ is a positive term,
- ϕ is a term (called a SAHLQVIST *antecedent*) constructed from boxed atoms, constants and negative terms with *lattice* operators of $\mathcal{L}_{MMV_d}$ (that includes $\odot$, $\oplus$, $\vee$ and $\wedge$).

Note that we allow to construct SAHLQVIST antecedents with MV-operators since these are lattice operators. Finally, note that we can equivalently replace the class of boxed atoms by the class of the *expanded box atoms* which are defined as the compositions of the terms $\tau_\oplus$ and $\tau_\odot$ preceded by a string of boxes. Indeed, any expanded boxed atom is equivalent to a boxed atom in the variety of MV-algebras with $\mathcal{L}$-operators.

To prove our SAHLQVIST equivalent, we follow the track proposed in [**58**]. The results are indeed easily adaptable to our many-valued realm.

LEMMA 4.39. *Assume that A is a $\mathcal{DLE}_{\mathcal{L}}$ and that t is an $\mathcal{L}$-term. If every operation symbol that occurs in t is interpreted as a lattice operator on A then t is stable on A.*

PROOF. Proposition 4.24 proves that the term t is expanding on A. Let us prove by induction on the number of operation symbols that occur in t that t is contracting on A. The base case is trivial. Let us then assume that $t = s(u_1, \ldots, u_k)$ where s is an operation symbol that is interpreted as a lattice operator on A and where $u_1, \ldots u_k$ are terms constructed with connectives that are interpreted as lattice operators on A. It follows that

$$t^{A^\sigma} = (s^A)^\sigma \circ (u_1^{A^\sigma}, \ldots, u_k^{A^\sigma}) \leq (s^A)^\sigma \circ ((u_1^A)^\sigma, \ldots, (u_k^A)^\sigma)$$

thanks to induction hypothesis. The map $(s^A)^\sigma$ is $(\iota^\uparrow, \iota^\uparrow)$-continuous since s^A is a lattice operator. Similarly, the map $(u_i^A)^\sigma$ is $(\sigma^\uparrow, \iota^\uparrow)$-continuous for any i in $\{1, \ldots, k\}$ since u_i^A is isotone. Consequently, tha map

$$(s^A)^\sigma \circ ((u_1^A)^\sigma, \ldots, (u_k^A)^\sigma)$$

turns out to be $(\sigma^\uparrow, \iota^\uparrow)$ continuous. The result then follows from the second item of Proposition 4.22. $\square$

LEMMA 4.40. *Let A be a $\mathcal{DLE}_\mathcal{L}$ and t be a term. If $t = s(u_1, \ldots, u_k)$ where for every operation symbol f that occurs in s, the map f^A is a lattice operator and where all the connectives in each of the u_i are $\wedge$-preserving operation on A, then τ is stable on A.*

PROOF. From Proposition 4.24, we deduce that t^A is expanding. Let us prove that it is contracting. We have

$$t^{A^\sigma} = s^{A^\sigma} \circ (u_1^{A^\sigma}, \ldots, u_k^{A^\sigma}) = (s^A)^\sigma \circ ((u_1^A)^\sigma, \ldots, (u_k^A)^\sigma),$$

thanks to the two preceding lemmas. Then, since each of the u_i is $(\sigma^\uparrow, \iota^\uparrow)$-continuous and since $(s^A)^\sigma = s^{A^\sigma}$ is $(\iota^\uparrow, \iota^\uparrow)$-continuous, we obtain that $(s^A)^\sigma \circ ((u_1^A)^\sigma, \ldots, (u_k^A)^\sigma)$ is $(\sigma^\uparrow, \iota^\uparrow)$-continuous and so that

$$(s^A)^\sigma \circ ((u_1^A)^\sigma, \ldots, (u_k^A)^\sigma) \leq (s^A(u_1^A, \ldots, u_k^A))^\sigma$$

thanks to the second item of Proposition 4.22. $\square$

The preceding developments lead us to the canonicity of SAHLQVIST equations.

THEOREM 4.41. *Every* SAHLQVIST *equation is canonical over the variety $\mathcal{MVO}_n^\mathcal{L}$.*

PROOF. We first consider the case of an equation $\phi(\beta_1, \ldots, \beta_k) \leq \psi$ where ψ is a positive term, the β_i's are boxed atoms and ϕ is constructed only with lattice operators (that includes $\vee, \wedge, \oplus, \odot$ and MV-operators).

Let A be an algebra of $\mathcal{MVO}_n^\mathcal{L}$. With the help of the preceding lemma, we obtain that $\phi(\beta_1, \ldots, \beta_k)$ is stable on A. Now, according to Propositions 4.37 and 4.24, the term ψ is (equivalent to) an expanding term on A. That is enough to conclude that the term $\phi(\beta_1, \ldots, \beta_k) \to \psi$ is stable on A.

Then, consider any SAHLQVIST equation

$$\phi(\beta_1, \ldots, \beta_k, \psi'_1, \ldots, \psi'_q) \leq \psi'$$

where the β_i's and ϕ are as above, the ψ'_i are negative and ψ' is a positive term. This equation is equivalent to

$$\neg\psi' \odot \phi(\beta_1, \ldots, \beta_k, \psi'_1, \ldots, \psi'_q) = 0.$$

Hence, any SAHLQVIST equation is equivalent to an equation of the kind

$$\phi(\beta_1, \ldots, \beta_k, \neg\psi_1, \ldots, \neg\psi_q) = 0$$

where ϕ and the β_i's as are above and the ψ_i belongs to Ψ. Since ϕ is isotone, this equation is in turn equivalent to the quasi-equation

$$(x_1 \leq \neg\psi_1, \ldots, x_q \leq \neg\psi_q) \Rightarrow \phi(\beta_1, \ldots, \beta_k, x_1, \ldots, x_q) = 0$$

where the x_i are new variables or, equivalently, to

$$(x_1 \odot \psi_1 = 0, \ldots, x_q \odot \psi_q = 0) \Rightarrow \phi(\beta_1, \ldots, \beta_k, x_1, \ldots, x_q) = 0.$$

We now introduce a new *lattice* operator E in the language and interpret it as the global modality:

$$E^A(x) = \begin{cases} 1 \text{ if } x > 0, \\ 0 \text{ if } x = 0. \end{cases}$$

Then, the latter quasi-equation is equivalent to the equation

$$\phi(\beta_1, \ldots, \beta_k, x_1, \ldots, x_q) \leq E(x_1 \odot \psi_1) \vee \cdots \vee E(x_q \odot \psi_q)$$

which is an equation that has already been proved to be canonical. $\square$

If we apply Proposition 4.33 to the preceding theorem, we obtain the following completeness result.

PROPOSITION 4.42. *Assume that $\mathcal{L}$ is a many-valued modal language. If ϕ is a formula constructed only with $\vee, \wedge, \oplus, \odot$ and modalities, if the term associated to ψ is positive and if β_i is a boxed atom or a formula whose associated term is negative for any i in $\{1, \ldots, k\}$ then $\mathbf{K}_n + \phi(\beta_1, \ldots, \beta_k) \to \psi$ is a* KRIPKE *complete logic.*

The reader may note that surprisingly, to obtain Proposition 4.42, we had to temporarily allow lattice (non MV-)operators in the language $\mathcal{L}$.

EXAMPLE 4.43. The equation $x \oplus x = x$ is canonical since it is equivalent to the SAHLQVIST equation $x \oplus x \leq x$. So, the logic $\mathbf{K}_n + (p \oplus p) \leftrightarrow p$ is canonical. It is easy to see that this logic is equal to $\mathbf{K}_1$ and hence, is not strongly KRIPKE complete.

Similarly, the equation $\Box(x \oplus x) \leq \Box x$ is a SAHLQVIST equation. Hence, the logic $\mathbf{K}_n + \Box(p \oplus p) \to \Box p$ is canonical. Actually, this is the logic that we have considered in Example 3.38. It is indeed easy to realize that for any Ł$_n$-valued frame $\mathfrak{F}$, we have $\mathfrak{F} \models \Box(p \oplus p) \to \Box p$ if and only if $\mathfrak{F} \models \Box(p \vee \neg p)$. Hence, the formula $\Box(p \vee \neg p)$ is valid in the canonical frame of $\mathbf{K}_n + \Box(p \oplus p) \to \Box p$ and belongs to $\mathbf{K}_n + \Box(p \oplus p) \to \Box p$. Conversely, the canonical frame of $\mathbf{K}_n + \Box(p \vee \neg p)$ validates the formula $\Box(p \oplus p) \to \Box p$, so that $\Box(p \oplus p) \to \Box p$ belongs to $\mathbf{K}_n + \Box(p \vee \neg p)$.

3. Strong canonical extensions

Any strongly canonical logic $\mathbf{L}$ is strongly complete. In this section, we develop the algebraic counterpart of strong canonicity. As already announced, a strong canonical logic will correspond to a strongly canonical variety.

The construction of the strong canonical extension of an MV$_n$-algebra is more dependent on the many-valued nature of A than on its lattice nature (unlike the construction of its canonical extension). But, the strong canonical extension of an MV$_n$-algebra and its canonical extension have, up to isomorphism, some important subalgebra in common, namely their algebra of idempotent elements.

We use this property to extend maps between MV$_n$-algebras to maps between their strong canonical extension. This will provide the strong canonical extension of an $\mathcal{MMV}_n$-algebra.

Our goal is Theorem 4.61 which is a SHALQVIST theorem to generate strongly canonical varieties.

DEFINITION 4.44. If A is an MV$_n$-algebra, we denote by A^τ the *strong canonical extension of* A, *i.e.*, the product MV$_n$-algebra Ł$_n^{A_{+n}}$.

LEMMA 4.45. *If A is an MV_n-algebra, then*

1. *the algebra A^τ is an MV_n-algebra and A^τ is an extension of A^σ which is an extension of A,*
2. *the lattice reduct of A^τ is a $\mathcal{DL}^+$,*
3. *the boolean algebras $\mathfrak{B}(A^\tau)$ and $\mathfrak{B}(A^\sigma)$ are isomorphic by a unique isomorphism that fixes $\mathfrak{B}(A)$,*
4. *if B is a complete and completely distributive MV_n-algebra that is an extension of A such that $\mathfrak{B}(B)$ is isomorphic to $\mathfrak{B}(A^\sigma)$ by a necessarily unique isomorphism $l : \mathfrak{B}(B) \to \mathfrak{B}(A^\sigma)$ fixing $\mathfrak{B}(A)$, then there is a unique embedding $\phi : B \to A^\tau$ that fixes the elements of $\mathfrak{B}(A^\sigma)$ (up to the isomorphism l and the isomorphism of item (3)).*

PROOF. The proofs of (1), (2), (3) are easy. The existence of the map ϕ in (4) can be obtained by carefully composing the various maps in game.

Let us prove that this map is unique. Assume that ψ satisfies the desired conditions. Then, for any x in B, the element $\psi(x)$ is fully determined by the element $(\tau_{1/n}(\psi(x)), \ldots, \tau_{n/n}(\psi(x)))$ of $(\mathfrak{B}(A^\tau))^n$. Now, for any i in $\{1, \ldots, n\}$, we have $\tau_{i/n}(\psi(x)) = \psi(\tau_{i/n}(x)) = l(b)$. Thus, the equality of ψ and ϕ follows from the fact that l is unique. □

The last item of the preceding lemma means that the strong canonical extension of an MV_n-algebra A can be defined, up to isomorphism, as the maximal extension of A that is a complete and completely distributive MV_n-algebra and whose boolean algebra of idempotents is isomorphic to the canonical extension of the boolean algebra of idempotents of A.

LEMMA 4.46. *If $A_1, \ldots, A_k$ are MV_n-algebras then $(A_1 \times \cdots \times A_k)^\tau = A_1^\tau \times \cdots \times A_k^\tau$.*

PROOF. The proof is direct. □

We now introduce a way to extend maps between two MV_n-algebras to maps between their strong canonical extensions. Unfortunately, the definition we adopt will not provide an extension for *any* map.

Recall that in an MV_n-algebra A, any element x is completely determined by the n-uple $(\tau_{1/n}(x), \ldots, \tau_{n/n}(x))$ of elements of $\mathfrak{B}(A)$. Hence, if A and B are two MV-algebras and if $f' : \mathfrak{B}(A) \to \mathfrak{B}(B)$ is a map, then we can define a map $f : A \to B$ by defining f as the unique map that satisfies $\tau_{i/n}(f(x)) = f'(\tau_{i/n}(x))$ for any x in A and any i in $\{1, \ldots, n\}$.

This is the track we follow to define an extension $f^\tau : A^\tau \to B^\tau$ of a map $f : A \to B$ between two MV_n-algebras A and B. According to our track, the building block of the extension is a map $f' : \mathfrak{B}(A^\tau) \to \mathfrak{B}(B^\tau)$. Since $\mathfrak{B}(A^\tau)$ is isomorphic to $\mathfrak{B}(A^\sigma)$ and to $\mathfrak{B}(A)^\sigma$, we may ride on the existing construction of canonical extension and want to define f' as one of the maps $(f\restriction_{\mathfrak{B}(A)})^\sigma$ or $f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$. Of course, in either case, the proposed map f' has to be valued in $\mathfrak{B}(B^\tau)$. A natural way to achieve this condition is to ensure that

$$\forall\, x \in A, f(x \oplus x) = f(x) \oplus f(x) \tag{3.1}$$

for the first case and that

$$\forall x \in A^\sigma, f^\sigma(x \oplus x) = f^\sigma(x) \oplus f^\sigma(x) \tag{3.2}$$

for the second case. Condition (3.2) implies obviously (3.1). Now, if $(\tau_\oplus \circ f)^\sigma = \tau_\oplus^\sigma \circ f^\sigma$ and $(f \circ \tau_\oplus)^\sigma = f^\sigma \circ \tau_\oplus^\sigma$, then, for any map f that satisfies (3.1)

$$f^\sigma \circ \tau_\oplus^\sigma = (f \circ \tau_\oplus)^\sigma = (\tau_\oplus \circ f)^\sigma = \tau_\oplus^\sigma \circ f^\sigma$$

and so (3.1) implies (3.2). We are so naturally lead to a problem about composition of canonical extensions that can be solved thanks to the tools that we have previously developed.

Lemma 4.47. *Assume that $f : A \to B$ is a map between two MV_n-algebras A and B.*

(1) *The identity $\tau_\oplus^\sigma \circ f^\sigma = (\tau_\oplus \circ f)^\sigma$ and the inequality $f^\sigma \circ \tau_\oplus^\sigma \leq (f \circ \tau_\oplus)^\sigma$ are satisfied.*
(2) *If f is an isotone map then the inequality $f^\sigma \circ \tau_\oplus^\sigma \geq (f \circ \tau_\oplus)^\sigma$ is satisfied.*

Proof. The identity $\tau_\oplus^\sigma \circ f^\sigma = (\tau_\oplus \circ f)^\sigma$ is a consequence of item (4) of Proposition 4.22. The inequality $f^\sigma \circ \tau_\oplus^\sigma \leq (f \circ \tau_\oplus)^\sigma$ is an application of item (5) of the same proposition. The last inequation is a consequence of item (2) of this proposition since $\tau_\oplus^\sigma$ is (σ, σ)-continuous and f^σ is $(\sigma, \iota^\uparrow)$-continuous. □

Recall that the map $f^\tau : A^\tau \to B^\tau$ that we want to define has to be an extension of f. The following lemma states that our methods of construction of f^τ provide an extension of f only if f commutes with $\tau_\oplus$ and $\tau_\odot$.

Lemma 4.48. *Assume that $f : A \to B$ is a map between two MV_n-algebras A and B.*

(1) *If $f^\sigma(\mathfrak{B}(A^\sigma)) \subseteq \mathfrak{B}(B^\sigma)$ and if $f' : A^\tau \to B^\tau$ denotes the map defined by $\tau_{i/n}(f'(x)) = f^\sigma\restriction_{\mathfrak{B}(A^\sigma)} (\tau_{i/n}(x))$ for any x in A^τ and any i in $\{1, \dots, n\}$ then $f'\restriction_A = f$ if and only if $f(\tau_{i/n}(x)) = \tau_{i/n}(f(x))$ for any i in $\{1, \dots, n\}$.*
(2) *If $f(\mathfrak{B}(A)) \subseteq \mathfrak{B}(B)$ and if $f' : A^\tau \to B^\tau$ denotes the map defined by $\tau_{i/n}(f'(x)) = f\restriction_{\mathfrak{B}(A)}^\sigma (\tau_{i/n}(x))$ for any x in A^τ and any i in $\{1, \dots, n\}$ then $f'\restriction_{\mathfrak{B}(A)} = f$ if and only if $f(\tau_{i/n}(x)) = \tau_{i/n}(f(x))$ for any i in $\{1, \dots, n\}$.*
(3) *The map f satisfies $f(\tau_{i/n}(x)) = \tau_{i/n}(f(x))$ for any i in $\{1, \dots, n\}$ if and only if $f(x \odot x) = f(x) \odot f(x)$ and $f(x \oplus x) = f(x \oplus x)$ for any x in A.*

Proof. (1) First assume that $f'\restriction_A = f$. If x belongs to A and i belongs to $\{1, \dots, n\}$, then $\tau_{i/n}(x)$ belongs to $\mathfrak{B}(A)$ and we obtain that

$$\tau_{i/n}(f(x)) = \tau_{i/n}(f'(x)) = f^\sigma\restriction_{\mathfrak{B}(A^\sigma)} (\tau_{i/n}(x)) = f(\tau_{i/n}(x))$$

since $f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$ is an extension of $f\restriction_{\mathfrak{B}(A)}$.

Conversely, if $f(\tau_{i/n}(x)) = \tau_{i/n}(f(x))$ for any x in A and any i in $\{1, \dots, n\}$ then if x belongs to A and i to $\{1, \dots, n\}$,

$$\tau_{i/n}(f'(x)) = f^\sigma\restriction_{\mathfrak{B}(A^\sigma)} (\tau_{i/n}(x)) = f(\tau_{i/n}(x)) = \tau_{i/n}(f(x)).$$

Thus, $f(x)$ and $f'(x)$ are equal.

(2) We proceed in a similar way.

(3) The right to left part of the statement is clear. For the left to right part we note that $f(\mathfrak{B}(A)) \subseteq \mathfrak{B}(B)$ and that $(f \circ \tau_\oplus)\restriction_{\mathfrak{B}(A)} = (\tau_\oplus \circ f)\restriction_{\mathfrak{B}(A)}$ since $\tau_\oplus\restriction_{\mathfrak{B}(C)}$ is the identity map for any MV-algebra C. We conclude that $f \circ \tau_\oplus = \tau_\oplus \circ f$ thanks to Proposition 3.15. We proceed in a similar way to prove that $f \circ \tau_\odot = \tau_\odot \circ f$. □

The preceding lemmas give a justification to the following definition.

DEFINITION 4.49. A map $f : A \to B$ between two MV-algebras A and B is an *idemorphism* if $f(x \oplus x) = f(x) \oplus f(x)$ and $f(x \odot x) = f(x) \odot f(x)$ for any x in A.

Let us sum up briefly the results we have obtained for the construction of f^τ. We want to ride on a map $f' : \mathfrak{B}(A^\tau) \to \mathfrak{B}(B^\tau)$ to define an extension $f^\tau : A^\tau \to B^\tau$ of a map $f : A \to B$. We have identified two candidates for the map f'. These candidates are $f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$ and $f\restriction^\sigma_{\mathfrak{B}(A)}$. In both cases, the map f^τ is an extension of f if and only if f is an idemorphism. We now prove that in that case, if further more f is isotone, then we do not have to choose between $f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$ and $f\restriction^\sigma_{\mathfrak{B}(A)}$.

LEMMA 4.50. *If $f : A \to B$ is an idemorphism between two MV_n-algebras A and B such that $f^\sigma(x \oplus^\sigma x) = f^\sigma(x) \oplus^\sigma f^\sigma(x)$ for any x in A^σ, then $(f\restriction_{\mathfrak{B}(A)})^\sigma = f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$.*
Consequently, if $f : A \to B$ is an isotone idemorphism, then $(f\restriction_{\mathfrak{B}(A)})^\sigma = f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$.

PROOF. We already know that

$$f^\sigma\restriction_{\mathfrak{B}(A^\sigma)} : \langle \mathfrak{B}(A^\sigma), \sigma(A^\sigma)\restriction_{\mathfrak{B}(A^\sigma)}\rangle \to \langle \mathfrak{B}(B^\sigma), \iota^\uparrow(B^\sigma)\restriction_{\mathfrak{B}(B^\sigma)}\rangle$$

is continuous. Up to the isomorphism and homeomorphism ϕ of Corollary 4.28, it means that the map

$$f^\sigma\restriction_{\mathfrak{B}(A^\sigma)} : \langle \mathfrak{B}(A)^\sigma, \sigma(\mathfrak{B}(A)^\sigma)\rangle \to \langle \mathfrak{B}(B)^\sigma, \iota^\uparrow(\mathfrak{B}(B)^\sigma)\rangle$$

is continuous. We conclude that $f^\sigma\restriction_{\mathfrak{B}(A^\sigma)} \leq (f\restriction_{\mathfrak{B}(A)})^\sigma$ since $(f\restriction_{\mathfrak{B}(A)})^\sigma$ is the largest extension of $(f\restriction_{\mathfrak{B}(A)})$ to $\mathfrak{B}(A)^\sigma$ that enjoys this property of continuity.

To obtain the other inequality, let us define the map $g : A^\sigma \to B^\sigma$ by setting $g(x) = y$ if $f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x)) = \tau_{i/n}(y)$. Of course, the maps g and $f\restriction^\sigma_{\mathfrak{B}(A)}$ coincide on $\mathfrak{B}(A^\sigma)$. Then, if we prove that g is $(\sigma, \iota^\uparrow)$-continuous, we will obtain that $g \leq f^\sigma$ on $\mathfrak{B}(A^\sigma)$ so that $f\restriction^\sigma_{\mathfrak{B}(A)} \leq f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$.

Let us prove that g is $(\sigma, \iota^\uparrow)$-continuous. Assume that p belongs to $J^\infty_\omega(B^\sigma)$. We obtain successively that

$$\begin{aligned} g^{-1}([p)) &= \{x \mid g(x) \geq p\} \\ &= \bigcap\{\{x \mid \tau_{i/n}(g(x)) \geq \tau_{i/n}(p)\} \mid i \in \{1, \ldots, n\}\} \\ &= \bigcap\{\{x \mid f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x)) \geq \tau_{i/n}(p)\} \mid i \in \{1, \ldots, n\}\} \\ &= \bigcap\{\tau^{-1}_{i/n}(f\restriction^{\sigma-1}_{\mathfrak{B}(A)} ([\tau_{i/n}(p)))) \mid i \in \{1, \ldots, n\}\}. \end{aligned}$$

Then, since $\tau_{i/n}(p)$ belongs to $J^\infty_\omega(\mathfrak{B}(A^\sigma)) = J^\infty_\omega(\mathfrak{B}(A)^\sigma)$, we can deduce from the $(\sigma, \iota^\uparrow)$-continuity of $f\restriction^\sigma_{\mathfrak{B}(A)}$ that $f\restriction^{\sigma-1}_{\mathfrak{B}(A)} ([\tau_{i/n}(p)))$ is an open of $\sigma(\mathfrak{B}(A)^\sigma)$. The conclusion then follows from the fact that the map $\tau^{A^\sigma}_{i/n} = (\tau^A_{i/n})^\sigma$ is $(\sigma(A^\sigma), \sigma(\mathfrak{B}(A)^\sigma))$-continuous since $\tau^A_{i/n}$ is both meet and join preserving. $\square$

In the applications we develop, the maps that we consider are isotone. Thus, we no not have to bother to distinguish $f\restriction^\sigma_{\mathfrak{B}(A)}$ from $f^\sigma\restriction_{\mathfrak{B}(A^\sigma)}$.

DEFINITION 4.51. Assume that $f : A \to B$ is an idemorphism between two MV_n-algebras A and B. The map $f^\tau : A^\tau \to B^\tau$ is defined by

$$\forall\, i \in \{1, \ldots, n\},\ \tau^{B^\tau}_{i/n}(f^\tau(x)) = f\restriction^\sigma_{\mathfrak{B}(A)} (\tau^{A^\tau}_{i/n}(x)),$$

and is called the *strong canonical extension of f*

LEMMA 4.52. *Assume that $f : A \to B$ is an idemorphism between two MV_n-algebras A and B. Then f^τ is an idemorphism. If f is an isotone map, a lattice operator, a dual lattice operator, a join preserving map or a meet preserving map then f^τ is a lattice operator, a dual lattice operator, a join preserving map, a meet preserving map respectively.*

PROOF. These results are proved in a similar way. We provide the proof for an idemorphism and a lattice operator.

If f is an idemorphism and if x is an element of A^τ then for any i in $\{1, \ldots, n\}$ we obtain successively, if l denotes the real $\min[\frac{i}{2n}, 1] \cap Ł_n$,

$$\begin{aligned}\tau_{i/n}(f^\tau(x \oplus x)) &= f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x \oplus x)) \\ &= f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_l(x)),\end{aligned}$$

and

$$\begin{aligned}\tau_{i/n}(f^\tau(x) \oplus f^\tau(x)) &= \tau_l(f^\tau(x)) \\ &= f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_l(x)).\end{aligned}$$

We follow that line of argument to prove that $f^\tau(x \odot x) = f^\tau(x) \odot f^\tau(x)$.

Let us then assume that $f : A_1 \times \cdots \times A_k \to B$ is an idemorphism and a lattice operator. We prove that f^τ respects the join on the first argument. If x_1 and x_1' belong to A_1 and if $(x_2, \ldots, x_k)$ belongs to $A_2 \times \cdots \times A_k$ then for any i in $\{1, \ldots, n\}$

$$\begin{aligned}\tau_{i/n}(f^\tau(x_1 \vee x_1', x_2, \ldots, x_k)) &= f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x_1 \vee x_1', x_2, \ldots, x_k)) \\ &= f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x_1 \vee x_1'), \ldots, \tau_{i/n}(x_k)) \\ &= f\restriction^\sigma_{\mathfrak{B}(A)} ((\tau_{i/n}(x_1) \vee \tau_{i/n}(x_1'), \ldots, \tau_{i/n}(x_k)))\end{aligned}$$

and we finally obtain that $\tau_{i/n}(f^\tau(x_1 \vee x_1', x_2, \ldots, x_k))$ is equal to

$$f\restriction^\sigma_{\mathfrak{B}(A)} ((\tau_{i/n}(x_1), \ldots, \tau_{i/n}(x_k))) \vee f\restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x_1'), \ldots, \tau_{i/n}(x_k)))$$

since $f\restriction^\sigma_{\mathfrak{B}(A)}$ is a lattice operator. This last element is in turn equal to

$$\tau_{i/n}(f^\tau(x_1, \ldots, x_k)) \vee \tau_{i/n}(f^\tau(x_1', \ldots, x_k)) = \tau_{i/n}(f^\tau(x_1, \ldots, x_k) \vee f^\tau(x_1', \ldots, x_k))$$

thanks to the definition of f^τ. □

EXAMPLE 4.53. If A is an MV$_n$-algebra then $\vee^A : A \times A \to A$ and $\wedge^A : A \times A \to A$ are two isotone idemorphisms. It is not hard to check that $\vee^\tau = \vee^{A^\tau}$ and that $\wedge^\tau = \wedge^{A^\tau}$.

Let us also remark that it is possible to consider the negation $\neg$ as an idemorphism. To do so, we need to consider $\neg$ as the map $\neg : A^\alpha \to A$. Then, we can prove that the map $\neg^\tau : A^{\tau\alpha} \to A^\tau$ is equal to the map $\neg^{A^\tau}$. Indeed, the map $\neg^\tau : A^{\tau\alpha} \to A^\tau$ is defined for every x in $A^{\tau\alpha}$ by

$$\tau^{A^\tau}_{i/n}(\neg^\tau x) = \neg\restriction^\sigma_{\mathfrak{B}(A)^\alpha} (\tau^{A^{\tau\alpha}}_{i/n}(x)) \quad \forall\, i \in \{i, \ldots, n\}.$$

Then, it follows successively that

$$\begin{aligned}\neg^\sigma(\tau^{A^{\tau\alpha}}_{i/n}(x)) &= \neg^{A^{\sigma\alpha}}(\tau^{A^{\tau\alpha}}_{i/n}(x)) \\ &= \neg^{A^{\tau\alpha}}(\tau^{A^{\tau\alpha}}_{i/n}(x)) \\ &= \tau^{A^\tau}_{i/n}(\neg^{A^\tau}(x)).\end{aligned}$$

PROPOSITION 4.54. *If ∇ is a k-ary dual MV-operator on an MV_n-algebra A then for any $(\alpha_1, \ldots, \alpha_k)$ in $(A^\tau)^k$ and any u in A_+*

$$\nabla^\tau(\alpha_1, \ldots, \alpha_k)(u) = \bigwedge_{\bar{v} \in Ru} (\alpha_1(v_1) \vee \cdots \vee \alpha_k(v_k))$$

where R denotes the canonical relation associated to ∇. Consequently, the map ∇^τ is a dual MV-operator.

PROOF. Assume that $(\alpha_1, \ldots, \alpha_k)$ belongs to $(A^\tau)^k$ and u belongs to A_+. For any i in $\{1, \ldots, n\}$ we obtain successively since ∇ is isotone that

$$\begin{aligned}
(\tau_{i/n}(\nabla^\tau(\alpha_1, \ldots, \alpha_k)))(u) &= (\nabla^\sigma(\tau_{i/n}(\alpha_1), \ldots, \tau_{i/n}(\alpha_k)))(u) \\
&= \bigwedge_{\bar{v} \in Ru} (\tau_{i/n}(\alpha_1)(\bar{v}_1) \vee \cdots \vee \tau_{i/n}(\alpha_k)(\bar{v}_k)) \\
&= \tau_{i/n}(\bigwedge_{\bar{v} \in Ru} (\alpha_1(\bar{v}_1) \vee \cdots \vee \alpha_k(\bar{v}_k))).
\end{aligned}$$

We then obtain that ∇^τ is a dual MV-operator thanks to Lemma 2.34 for example. □

It is now time to give results about composition of τ-extensions. Once again, our results follow from the results about composition of canonical extensions.

PROPOSITION 4.55. *Assume that $f : B \to C$ and $g : A \to B$ are two idemorphisms between the MV_n-algebras A, B and C. If $\bowtie$ belongs to $\{\leq, \geq, =\}$ and if $(fg) \restriction^\sigma_{\mathfrak{B}(A)} \bowtie f \restriction^\sigma_{\mathfrak{B}(A)} g \restriction^\sigma_{\mathfrak{B}(A)}$ then $(fg)^\tau \bowtie f^\tau g^\tau$.*

PROOF. Assume that $(fg) \restriction^\sigma_{\mathfrak{B}(A)} \bowtie f \restriction^\sigma_{\mathfrak{B}(A)} g \restriction^\sigma_{\mathfrak{B}(A)}$. If x belongs to A^τ and i belongs to $\{1, \ldots, n\}$, we obtain successively

$$\begin{aligned}
\tau_{i/n}((fg)^\tau(x)) &= (fg) \restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x)) \\
&\bowtie f \restriction^\sigma_{\mathfrak{B}(A)} (g \restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(x))) \\
&= f \restriction^\sigma_{\mathfrak{B}(A)} (\tau_{i/n}(g^\tau(x))) \\
&= \tau_{i/n}(f^\tau g^\tau(x)),
\end{aligned}$$

which concludes the proof. □

In order to determine if a variety $\mathcal{A}$ of MV$_n$-algebras with $\mathcal{L}$-operators is closed under taking τ-extensions, it is useful to prove that if B is a quotient of the $\mathcal{A}$-algebra A, then B^τ is a quotient of A^τ. We first consider the more general problem of the conservation of homomorphisms: if $f : A \to B$ is an homomorphism between two $\mathcal{A}$-algebras A and B, can we deduce that $f^\tau : A^\tau \to B^\tau$ is an homomorphism?

We have to take care that, unlike the case of canonical extension, the operation $\oplus^{A^\tau}$ is not obtained as the τ-extensions $\oplus^A$ since it is not an idemorphism.

The result we obtain is more general than needed.

DEFINITION 4.56. An algebra A is an *MV_n-algebra with $\mathcal{L}$-idemorphisms* (resp. *MV_n-algebra with $\mathcal{L}$-lattice idemorphisms*) if it is an $\mathcal{L}$-algebra such that $\langle A, \oplus, \neg, 0, 1\rangle$ is an MV$_n$-algebra and if any operation g of $\mathcal{L} \setminus \mathcal{L}_{MV}$ is interpreted as an idemorphism (resp. and as a lattice operator) g^A on the MV-algebra reduct of A.

If A is an MV$_n$-algebra with $\mathcal{L}$-idemorphisms, the *strong canonical extension* A^τ of A is defined as the $\mathcal{L}$-algebra whose MV-reduct is the strong canonical extension of the MV-reduct of A and that satisfies $g^{A^\tau} = (g^A)^\tau$ for any operation symbol g in $\mathcal{L} \setminus \mathcal{L}_{MV}$.

So, in the construction of strong canonical extensions of MV$_n$-algebras, the algebras are considered more as expanded MV-algebras than expanded $\mathcal{DL}$s.

Examples of $\mathcal{L}$-algebras with lattice idemorphisms are given by $\mathcal{L}$-algebras with MV-operators.

LEMMA 4.57. *Assume that A and B are MV_n-algebras with $\mathcal{L}$-lattice idemorphisms. For any $\mathcal{L}$-homomorphism $f : A \to B$ the map $f^\tau : A^\tau \to B^\tau$ is an $\mathcal{L}$-homomorphism.*

PROOF. First, assume that g is a k-ary operation of $\mathcal{L} \setminus \mathcal{L}_{MV}$ interpreted as a lattice idemorphism on the algebras A^k and B^k. If $(x_1, \dots, x_k)$ belongs to $(A^\tau)^k$ and i belongs to $\{1, \dots, n\}$, we obtain successively on the one hand that

$$\begin{aligned}
\tau_{i/n}((f^\tau(g^A)^\tau(x_1,\dots,x_k))) &= (f\lfloor^{\sigma}_{\mathfrak{B}(A)}\, g^A\lfloor^{\sigma}_{\mathfrak{B}(A)^k})(\tau_{i/n}(x_1),\dots,\tau_{i/n}(x_k)) \\
&= (g^B\lfloor^{\sigma}_{\mathfrak{B}(B)^k}\, f\lfloor^{\sigma[k]}_{\mathfrak{B}(A)})(\tau_{i/n}(x_1),\dots,\tau_{i/n}(x_k)) \\
&= \tau_{\frac{i}{n}}((g^B)^\tau f^{\tau[k]}(x_1,\dots,x_k))
\end{aligned}$$

since we have showed in the proof of in Proposition 4.31 that the canonical extension of an $\mathcal{L}$-homomorphism respects isotone operations.

Let us now prove that $f^\tau(x \oplus^{A^\tau} y) = f^\tau(x) \oplus^{B^\tau} f^\tau(y)$ for any x and y in A^τ. Let i be an element of $\{1, \dots, n\}$. The equation

$$\tau_{i/n}(x \oplus y) = \tau_{i/n}(x) \vee (\tau_{(i-1)/n}(x) \wedge \tau_{1/n}(y)) \vee \cdots \vee (\tau_{1/n}(x) \wedge \tau_{(i-1)/n}(y)) \vee \tau_{i/n}(y) \tag{3.3}$$

(where we τ_0 is defined as the constant term 1) is satisfied in the variety of MV$_n$-algebras. If x and y belong to A^τ, then $\tau_{i/n}(f^\tau(x) \oplus f^\tau(y))$ is equal, thanks to equation (3.3), to

$$\tau_{i/n}(f^\tau(x)) \vee (\tau_{(i-1)/n}(f^\tau(x)) \wedge \tau_{1/n}(f^\tau(y))) \vee \cdots \vee (\tau_{1/n}((f^\tau(x)) \wedge \tau_{(i-1)/n}((f^\tau(y))) \vee \tau_{i/n}((f^\tau(y)),$$

which is in turn equal, by definition of f^τ, to

$$f^\sigma(\tau_{i/n}(x)) \vee (f^\sigma(\tau_{(i-1)/n}(x)) \wedge f^\sigma(\tau_{1/n}(y))) \vee \cdots \vee (f^\sigma(\tau_{1/n}(x)) \wedge f^\sigma(\tau_{(i-1)/n}(y))) \vee f^\sigma(\tau_{i/n}(y)).$$

Then, since $f^\sigma : A^\sigma \to B^\sigma$ is an homomorphism of MV-algebras, this last element is equal to

$$f^\sigma(\tau_{i/n}(x) \vee (\tau_{(i-1)/n}(x) \wedge \tau_{1/n}(y)) \vee \cdots \vee (\tau_{1/n}(x) \wedge \tau_{(i-1)/n}(y)) \vee \tau_{i/n}(y)),$$

i.e., to

$$f^\sigma(\tau_{i/n}(x \oplus y)) = \tau_{i/n}(f^\tau(x \oplus y)).$$

We proceed in a similar way to prove that $f^\tau(\neg x) = \neg f(x)$ for any x in A^τ. □

DEFINITION 4.58. A variety $\mathcal{A}$ of $\mathcal{L}$-algebras with MV-operators is *strongly canonical* if it contains the strong canonical extension of any of its algebras.

PROPOSITION 4.59. *Assume that* $\mathbf{L}$ *is an Ł$_n$-valued modal logic. The variety of* $\mathbf{L}$*-algebras is strongly canonical if and only if* $\mathbf{L}$ *is a strongly canonical logic.*

PROOF. Assume that the variety $\mathcal{A}_{\mathbf{L}}$ of $\mathbf{L}$-algebras is strongly canonical. Then, for any set X, the algebra $\mathcal{F}_{\mathbf{L}}(X)^{\tau}$ belongs to $\mathcal{A}_{\mathbf{L}}$, which means that $\mathcal{F}_{\mathbf{L}}(X)_{+} \models \mathbf{L}$.

Assume conversely that $\mathbf{L}$ is a strongly canonical logic. For any algebra A of $\mathcal{MMV}_{\mathbf{L}}$ there is a set X such that A is a quotient of $\mathcal{F}_{\mathbf{L}}(X)$. Since $\mathcal{F}_{\mathbf{L}}(X)_{+} \models \mathbf{L}$, we obtain that $\mathcal{F}_{\mathbf{L}}(X)^{\tau}$ belongs to $\mathcal{MMV}_{\mathbf{L}}$ and an application of Lemma 4.57 proves that A^{τ} also belongs to $\mathcal{MMV}_{\mathbf{L}}$. □

So, the preceding lemma provides a tool to obtain strongly KRIPKE-complete logics.

We can for example determine the equations that are strongly canonical (*i.e.*, define strongly canonical varieties) among SAHLQVIST equations.

LEMMA 4.60. *Assume that A is an MV_n-algebra with $\mathcal{L}$-operators.*

(1) *If t is an $\mathcal{L}$-term constructed with operations that are interpreted as isotone idemorphisms on A then $t^{A^{\tau}} \geq (t^{A})^{\tau}$.*
(2) *If t is an $\mathcal{L}$-term constructed with operations that are interpreted as lattice idemorphisms on A then $t^{A^{\tau}} = (t^{A})^{\tau}$.*
(3) *If $t = s(u_1, \ldots, u_k)$ is an $\mathcal{L}$-term where for every operation symbol f that appears in s the map f^{A} is a lattice idemorphism and where all the operations in each of the u_i are interpreted as meet preserving idemorphisms, then $(t^{A^{\tau}}) = (t^{A})^{\tau}$.*

PROOF. The proofs are done by induction on the number of connectives in t with the help of proposition 4.55 and the corresponding results for canonical extensions. □

THEOREM 4.61. *Assume that $\phi \leq \psi$ is a* SAHLQVIST *equation over the language $\mathcal{L}_{MMV^d}$ where*

- *the term ψ is constructed only with the operations $\neg$, $\vee$, $\wedge$, constants, modalities and dual modalities,*
- *the term ϕ is constructed from boxed atoms, constants with the operations $\vee$, $\wedge$ and modalities.*

The equation $\phi \leq \psi$ is strongly canonical and thus the logic $\mathbf{K}_n + \phi \to \psi$ is a KRIPKE*-complete logic.*

EXAMPLE 4.62. The equations $\Box p \to p$, $\Box p \to \Box\Box p$, $p \to \Box\Diamond p$ are all strongly canonical and hence define strongly KRIPKE complete logics.

CHAPTER 5

A topological duality for the category $\mathcal{MMV}_n^{\mathcal{L}}$

By adding a topological layer to the $Ł_n$-valued $\mathcal{L}$-frames, we are going to produce a category which is dually equivalent to the category of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras. This duality is an extension of the STONE duality for boolean algebras with $\mathcal{L}$-operators.

To construct this duality, we can follow two different but equivalent paths : we can add a topological ingredient to the existing structures or we can add structure to an existing duality for the variety of MV_n-algebras. Because this last path drives us to the desired result more quickly, we have decided to follow that one.

1. A natural duality for the algebras of ŁUKASIEWICZ $n+1$-valued logic

It is well known that a strong natural duality (in the sense of DAVEY and WERNER in [**11**]) can be constructed for each of the varieties $\mathcal{MV}_n = \mathbb{HSP}(Ł_n) = \mathbb{ISP}(Ł_n)$. The existence of these natural dualities is a consequence of the semi-primality of $Ł_n$. This fact was first noticed by CIGNOLI in [**9**] and the consequences of this duality were studied with more details in [**46**].

These dualities, from which we can recover the STONE duality for boolean algebras by considering $n = 1$, are a good starting point for the construction of a duality for the varieties of MV_n-algebras with $\mathcal{L}$-operators. Indeed, in the classical two-valued case, the dual of a boolean algebra with operators is obtained by adding a structure (a $k+1$-ary relation R for any k-ary boolean operator of $\mathcal{L}$) to the STONE dual of the boolean reduct of B. It is the idea we propose to follow in this section : the dual of an MV_n-algebra with $\mathcal{L}$-operators A will be obtained by adding a structure to the dual of the reduct of A in $\mathcal{MV}_n$.

We recall the basic facts about the natural duality for $\mathcal{MV}_n$. In order to improve the readability of this document, we do not follow strictly the notations that are in effect in the theory of natural duality (usually, algebras are denoted by underlined Roman capital letters and structures by "undertilded" Roman capital letters).

DEFINITION 5.1. We denote by $\utilde{Ł}_n$ the topological structure

$$\utilde{Ł}_n = \langle Ł_n; \{Ł_m \mid m \in \mathrm{div}(n)\}, \tau\rangle,$$

where τ is the discrete topology, $\mathrm{div}(n)$ is the set of the positive divisors of n and $Ł_m$ (with $m \in \mathrm{div}(n)$) is the subalgebra of $Ł_n$ viewed as a distinguished (closed) subspace of $\langle Ł_n, \tau\rangle$ (and can also be viewed as an unary relation on $Ł_n$).

We denote by $\mathcal{MV}_n$ the category whose objects are the members of the variety $\mathcal{MV}_n = \mathbb{HSP}(Ł_n) = \mathbb{ISP}(Ł_n)$ and whose morphisms are the MV-homomorphisms.

Finally, we denote by $\mathcal{X}_n$ the category whose objects are the members of the topological quasi-variety $\mathbb{IS}_c\mathbb{P}(\underset{\sim}{Ł}_n)$ (*i.e.*, the topological structures that are isomorphic to a closed substructure of a power of $\underset{\sim}{Ł}_n$) and whose morphisms are the continuous maps $\phi : X \to Y$ such that $\phi(r_m^X) \subseteq r_m^Y$.

If A is an $\mathcal{MV}_n$-algebra, the set $\mathcal{MV}_n(A, Ł_n)$ is viewed as a substructure of $Ł_n^A$ and is equipped with the topology induced by $Ł_n^A$. So, if $[a : \frac{i}{n}]$ denotes the subspace $\{u \in \mathcal{MV}_n(A, Ł_n) \mid u(a) = \frac{i}{n}\}$ whenever $a \in A$ and $i \in \{0, \dots, n\}$, then $\{[a : \frac{i}{n}] \mid a \in A \text{ and } i \in \{0, \dots, n\}\}$ is a clopen subbasis of the topology of $\mathcal{MV}_n(A, Ł_n)$. Note that it is also the case of $\{[b : 1] \mid b \in \mathfrak{B}(A)\}$.

The results about natural duality for $\mathcal{MV}_n$ can be briefly summarized by the following proposition (see [**46**]).

Proposition 5.2. *Let us denote by* D_n *and* E_n *the functors*

$$\mathsf{D}_n : \mathcal{MV}_n \to \mathcal{X}_n : \begin{cases} A \in \mathcal{MV}_n \mapsto \mathsf{D}_n(A) = \mathcal{MV}_n(A, Ł_n) \\ f \in \mathcal{MV}_n(A, B) \mapsto \mathsf{D}_n(f) \in \mathcal{X}_n(\mathsf{D}_n(B), \mathsf{D}_n(A)), \end{cases}$$

where $\mathsf{D}_n(f)(u) = u \circ f$ *for all* $u \in \mathsf{D}_n(B)$, *and*

$$\mathsf{E}_n : \mathcal{X}_n \to \mathcal{MV}_n : \begin{cases} X \in \mathcal{X}_n \mapsto \mathsf{E}_n(X) = \mathcal{X}_n(X, Ł_n) \\ \psi \in \mathcal{X}_n(X, Y) \mapsto \mathsf{E}_n(\psi) \in \mathcal{MV}_n(\mathsf{E}_n(Y), \mathsf{E}_n(X)), \end{cases}$$

where $\mathsf{E}_n(\psi)(\alpha) = \alpha \circ \psi$ *for all* $\alpha \in \mathsf{E}_n(Y)$.

The functors D_n *and* E_n *define a strong natural duality between the category* $\mathcal{MV}_n$ *and* $\mathcal{X}_n$. *Thus, these two functors map embeddings onto surjective morphisms and conversely.*

The canonical isomorphism between an $\mathcal{MV}_n$-algebra A and its bidual $\mathsf{E}_n\mathsf{D}_n(A)$ is the evaluation map

$$e_A : A \to \mathsf{E}_n\mathsf{D}_n(A) : a \mapsto e_A(a) : u \mapsto u(a),$$

and if X is an object of $\mathcal{X}_n$, the map

$$\epsilon_X : X \to \mathsf{D}_n\mathsf{E}_n(X) : u \mapsto \epsilon_X(u) : \alpha \mapsto \alpha(u)$$

is the canonical $\mathcal{X}_n$-isomorphism between X and $\mathsf{D}_n\mathsf{E}_n(X)$.

As we have already taken some liberty in our notations, if X is a structure of $\mathcal{X}_n$, we are going to denote by X the structure, the universe of the structure but also the underlying topological space of X. The context is always clear enough to suggest the right level behind such a notation.

The following result is a characterization of the objects of $\mathcal{X}_n$ (see [**46**] or [**11**]).

Proposition 5.3. *A structure*

$$X = \langle X; \{r_m^X \mid m \in \operatorname{div}(n)\}, \tau\rangle,$$

is an object of $\mathcal{X}_n$ *if and only if*

- (X1) $\langle X, \tau\rangle$ *is a* Boolean *space (*i.e., τ *is a compact* Hausdorff *zero-dimensional topology);*
- (X2) r_m^X *is a closed subspace of* X *for every* $m \in \operatorname{div}(n)$;
- (X3) $r_n^X = X$ *and* $r_m^X \cap r_k^X = r_{\gcd(m,k)}^X$ *for every* m *and* k *in* $\operatorname{div}(n)$.

If no confusion is possible, we prefer to denote by r_m the relation r_m^X. Note that if we consider $n = 1$, it is easy to realize that the duality for $\mathcal{MV}_1$ is equivalent to the well known STONE duality for boolean algebras (there is no need in that case to add the information provided by the subalgebras of $Ł_1$ since the only subalgebra of $Ł_1$ is $Ł_1$). We thus naturally denote by $\mathsf{D}_1(A)$ the STONE dual of any boolean algebra A.

LEMMA 5.4. *If A is an $\mathcal{MV}_n$-algebra then the underlying topological space of $\mathsf{D}_\mathsf{n}(A)$ is homeomorphic to $\mathsf{D}_1(\mathfrak{B}(A))$.*

PROOF. We work up to the canonical homeomorphism beetween $\mathsf{D}_1(\mathfrak{B}(A))$ and $\mathsf{D}_n(\mathfrak{B}(A))$. If $i : \mathfrak{B}(A) \to A$ denotes the inclusion map, then we prove that $\mathsf{D}_n(i) : \mathsf{D}_n(A) \to \mathsf{D}_1(\mathfrak{B}(A)) : u \mapsto u \circ i$ is an homeomorphism. Since i is an embedding, we already know that $\mathsf{D}_n(i)$ is a continuous onto map. If we prove that it is a one-to-one map, we can conclude that it is an homeomorphism since $\mathsf{D}_n(A)$ is compact and $\mathsf{D}_1(\mathfrak{B}(A))$ is HAUSDORFF.

Now, assume that u and v are two elements of $\mathsf{D}_n(A)$ such that $u \circ i = v \circ i$. Then, since $u \neq v$, there is an x in A and an i in $\{1, \ldots, n\}$ such that $u(x) < \frac{i}{n} \leq v(x)$. It follows that $u(\tau_{i/n}(x)) = 0$ and $v(\tau_{i/n}(x)) = 1$, which is a contradiction since $\tau_{i/n}(x)$ belongs to $\mathfrak{B}(A)$. $\square$

Eventually, note that other types of dualities have been considered for the variety of MV-algebras, *e.g.*, [**44, 8**].

2. Dualization of objects

We just have to add a topological layer to the canonical $Ł_n$-valued $\mathcal{L}$-frame associated to a $\mathcal{MMV}_n^{\mathcal{L}}$-algebra A in order to obtain a representation result of A as a concrete algebra of morphisms. We actually mix the topological duality for $\mathcal{MV}_n$ with the results about canonical entities associated to $\mathcal{MMV}_n^{\mathcal{L}}$-algebras.

DEFINITION 5.5. If A is an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra then for any k-ary dual modality ∇ of $\mathcal{L}$, we denote by $R_\nabla^{\mathsf{D}_n(A)}$ (or simply R_∇ if no confusion is possible) the canonical $k+1$-ary relation associated to ∇ in Definition 2.37.

That is, if we equip $\mathsf{D}_n(A)$ with the relations $R_\nabla^{\mathsf{D}_n(A)}$ for any dual modality ∇ of $\mathcal{L}$ and forget the topology of $\mathsf{D}_n(A)$, we obtain the canonical $Ł_n$-valued $\mathcal{L}$-frame associated to A. Thus, for any u in $\mathsf{D}_n(A)$ and any divisor m of n, if u belongs to $r_m^{\mathsf{D}_n(A)}$ and $(u, v_1, \ldots, v_k)$ belongs to R_Δ then v_i belongs to $r_m^{\mathsf{D}_n(A)}$ for any i in $\{1, \ldots, k\}$ (see Lemma 2.42). Moreover, for any k-ary dual MV-operator ∇ of $\mathcal{L}$, any $(x_1, \ldots, x_k)$ in A^k and any u in $\mathsf{D}_n(A)$, we obtain

$$u(\nabla(x_1, \ldots, x_k)) = \bigwedge_{\bar{v} \in R_\nabla u} (\bar{v}_1(x_1) \vee \cdots \vee \bar{v}_k(x_k)).$$

(see Propotisiom 2.40).

We now have to determine how the relations $R_\Delta^{\mathsf{D}_n(A)}$ interact with the topology of $\mathsf{D}_n(A)$.

PROPOSITION 5.6. *If A is an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra then for any k-ary dual MV-operator ∇ of $\mathcal{L}$,*

(1) *for any clopen subsets $\Omega_1, \ldots, \Omega_k$ of X, the set $R^{-1}(\Omega_1 \times \cdots \times \Omega_k)$ is a clopen subset of X,*
(2) *the relation $R_\nabla^{\mathsf{D}_n(A)}$ is a closed subspace of $\mathsf{D}_n(A)^{k+1}$,*

(3) *$R(r_m)$ is a subset of r_m^k.*

PROOF. To obtain (1) and (2), we can prove by combining Lemma 5.4 and Lemma 2.38 that the topological structures $\langle \mathsf{D}_n(A); \{R_{\nabla_i} \mid i \in I\}, \tau\rangle$ and $\langle \mathsf{D}_1(\mathfrak{B}(A)); \{R_{\nabla_i\downharpoonright_{\mathfrak{B}(A)}} \mid i \in I\}, \tau\rangle$ are isomorphic. So, the desired results are obtained thanks to corresponding results for the STONE duality for boolean algebras with $\mathcal{L}$-operators.

(3) The third statement is known (see Lemma 2.42). □

Recall that if R is a relation on boolean space X, then R is a closed relation if and only if Ru is a closed subspace of X for any u in X.

It turns out that these properties characterize relations that are dual to dual MV-operators.

DEFINITION 5.7. If X is a structure of $\mathcal{X}_n$, a subset R of X^{k+1} is a $k+1$-*ary modal relation* on X if

(1) R is a closed subspace of X^{k+1},
(2) for any clopen subsets $\Omega_1, \ldots, \Omega_k$ of X, the set $R^{-1}(\Omega_1 \times \cdots \times \Omega_k)$ is a clopen subset of X,
(3) for any positive divisor m of n, the set $R(r_m)$ is a subset of r_m^k.

We denote by $\mathcal{MX}_n^{\mathcal{L}}$ the class of the topological structures

$$\langle X, \{r_m \mid m \in \mathrm{div}(n)\}, \{R_{\Delta_i} \mid i \in I\}; \tau\rangle$$

such that $\langle X, \{r_m \mid m \in \mathrm{div}(n)\}, \tau\rangle$ is an object of $\mathcal{X}_n$ and R_{Δ_i} is a $k_i + 1$-ary modal relation on X for any i in I.

On the one hand, an $\mathcal{MX}_n^{\mathcal{L}}$-structure X can be considered as an Ł$_n$-valued $\mathcal{L}$-frame with a new topological layer. Let us temporally denote by $\mathfrak{F}_X$ the underlying Ł$_n$-valued $\mathcal{L}$-frame of X. On the other hand, the structure X can aslo be viewed as a topological structure of $\mathcal{X}_n$ on which more structure (defined by the modal relations) has been added. Let us temporally denote by $\underset{\sim}{X}$ the $\mathcal{X}_n$-reduct of X. We obviously obtain that the dual algebra $\mathsf{E}_n(\underset{\sim}{X})$ of $\underset{\sim}{X}$ is a subalgebra of the MV-reduct of the Ł$_n$-tight complex algebra $(\mathfrak{F}_X)^{\times_n}$ of $\mathfrak{F}_X$. The following result states that actually, the algebra $\mathsf{E}_n(\underset{\sim}{X})$ is more than that: it is also an $\mathcal{L}$-subalgebra of $(\mathfrak{F}_X)^{\times_n}$.

PROPOSITION 5.8. *If X is an $\mathcal{MX}_n^{\mathcal{L}}$-structure and if ∇_i denotes the dual MV-operator associated to R_i^X in Definition 2.33 for any i in I, then the algebra $\langle \mathsf{E}_n(X), \{\nabla_i \mid i \in I\}\rangle$ is a subalgebra of $(\mathfrak{F}_X)^{\times_n}$.*

PROOF. We have to prove that the map $\nabla_i(\alpha_1, \ldots, \alpha_k)$ is a continuous map from X to $\underset{\sim}{\text{Ł}}_n$ for any $\alpha_1, \ldots, \alpha_k$ of $\mathsf{E}_n(X)$. It appears clearly that

$$(\nabla_i(\alpha_1, \ldots, \alpha_k))^{-1}([\frac{i}{n}]) = (\nabla_i(\tau_{\frac{i}{n}}(\alpha_1), \ldots, \tau_{\frac{i}{n}}(\alpha_k)))^{-1}(1)$$

for any i in $\{1, \ldots, n\}$. Up to the homeomorphism of Lemma 5.4, we obtain that the element $\nabla_i(\tau_{\frac{i}{n}}(\alpha_1), \ldots, \tau_{\frac{i}{n}}(\alpha_k))$ is just a member of the bidual of the boolean algebra with $\mathcal{L}$-operators obtained by equipping $\mathfrak{B}(A)$ with $\nabla\downharpoonright_{\mathfrak{B}(A)}$ for any dual modality ∇ of $\mathcal{L}$. In this respect, the later map is continuous. □

Thus, for any X in $\mathcal{MX}_n^{\mathcal{L}}$, the algebra $\langle \mathsf{E}_n(X), \{\nabla_i \mid i \in I\}\rangle$ is an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra. The following result gives a representation of $\mathcal{MMV}_n^{\mathcal{L}}$-algebra as an algebra of morphisms.

PROPOSITION 5.9. *If A is an $\mathcal{MMV}_n^{\mathcal{L}}$-algebra then the evaluation map e_A is an $\mathcal{L}$-isomorphism from A to $\langle \mathsf{E}_n\mathsf{D}_n(A), \{\nabla_i \mid i \in I\}\rangle$.*

PROOF. As we already know that e_A is an isomorphism of MV-algebras, stating that e_A is an isomorphism of $\mathcal{MMV}_n^{\mathcal{L}}$-algebras is equivalent to stating that for any k-ary dual modality ∇ of $\mathcal{L}$ and any $(a_1, \dots, a_k)$ in A^k ,

$$u(\nabla(a_1, \dots, a_k)) = \bigwedge_{\bar{v} \in R_\nabla u} \bar{v}_1(a_1) \vee \dots \vee \bar{v}_k(a_k).$$

This is well known (see our remark following Definition 5.5) □

Here is the corresponding representation result for $\mathcal{MX}_n$-structures.

PROPOSITION 5.10. *Assume that X is an $\mathcal{MX}_n^{\mathcal{L}}$-structure. For any i in I, we denote by ∇_i the dual MV-operator on $\mathsf{E}_n(X)$ associated to R_i and by R_{∇_i} the relation on $\mathsf{D}_n\mathsf{E}_n(X)$ associated to ∇_i. Then, the relations R_i and R_{∇_i} coincides up the canonical $\mathcal{X}_n$-morphism ϵ_X.*

PROOF. Assume that R is a k-ary relation of $\{R_i \mid i \in I\}$. We have already proved that if $(u, v_1, \dots, v_k)$ belongs to R then $(\epsilon_X(u), \epsilon_X(v_1), \dots, \epsilon_X(v_k))$ belongs to R_{∇_i} (see 2.65, and there is no need of the topological layer fot this).

Assume now that $(\epsilon_X(u), \epsilon_X(v_1), \dots, \epsilon_X(v_k))$ belongs to R_{∇_i}, *i.e.*, assume that for any $\alpha_1, \dots, \alpha_k$ in $\mathsf{E}_n(X)$,

$$(\nabla_i(\alpha_1, \dots, \alpha_k))(u) = 1 \Rightarrow \alpha_1(v_1) \vee \dots \vee \alpha_k(v_k) = 1.$$

If $(u, v_1, \dots, v_k)$ does not belong to R, then there is a clopen subset Ω of X^{k+1} such that

$$(u, v_1, \dots, v_k) \in \Omega \subseteq X^{k+1} \setminus R. \tag{2.1}$$

Equivalently, there are elements $\alpha, \beta_1, \dots, \beta_k$ of $\mathfrak{B}(\mathsf{E}_n(X))$ such that

$$(u, v_1, \dots, v_k) \in \alpha^{-1}(0) \times \dots \times \beta_1^{-1}(0) \times \beta_k^{-1}(0) \subseteq X^{k+1} \setminus R. \tag{2.2}$$

We can thus conclude that Ru is a subset of $X^k \setminus (\beta_1^{-1}(0) \times \dots \times \beta_k^{-1}(0))$ and so that $\nabla_i(\beta_1, \dots, \beta_k)(u) = 1$. We deduce from (2.1) that there is a j in $\{1, \dots, k\}$ such that $\beta_j(v_j) = 1$, which contradicts (2.2). □

3. Dualization of morphisms

The previous section provides a representation theorem the $\mathcal{MMV}_n^{\mathcal{L}}$-algebras. We now introduce the right notion of morphism between $\mathcal{MX}_n$-structures in order to lift this representation result at a categorical level.

To define the suitable notion of morphism between $\mathcal{MX}_n^{\mathcal{L}}$-structures we proceed in a very natural way. Such a morphism $\phi : X \to Y$ is defined as a map that preserves both the $\mathcal{X}_n$-reduct of X and its Ł$_n$-valued $\mathcal{L}$-frame reduct. These morphisms are good candidates for the dualization of $\mathcal{MMV}_n^{\mathcal{L}}$-homomorphisms since, roughly speaking, the first condition dualizes the "modal-preserving" fragment of $\mathcal{MMV}_n^{\mathcal{L}}$-homomorphisms and the second one dualizes their "MV-preserving fragment".

DEFINITION 5.11. If X and Y are two $\mathcal{MX}_n^{\mathcal{L}}$-structures, a map $\phi : X \to Y$ is an $\mathcal{MX}_n^{\mathcal{L}}$-morphism if

(1) the map ϕ is an $\mathcal{X}_n$-morphism,
(2) the map ϕ is a bounded morphism between the underlying $\mathcal{L}$-frames of X and Y.

PROPOSITION 5.12. *If $f : A \to B$ is an $\mathcal{MMV}_n^{\mathcal{L}}$-homomorphism between two $\mathcal{MMV}_n^{\mathcal{L}}$-algebras A and B, then $\mathsf{D}_n(f) : \mathsf{D}_n(B) \to \mathsf{D}_n(A)$ is an $\mathcal{MX}_n$-morphism.*

PROOF. The proof is a combination of Proposition 5.2, Lemma 5.4 and Lemma 2.38. □

PROPOSITION 5.13. *If $\phi : X \to Y$ is an $\mathcal{MX}_n$-morphism between two $\mathcal{MX}_n^{\mathcal{L}}$-structures X and Y then $\mathsf{E}_n(\phi)$ is an $\mathcal{MMV}_n^{\mathcal{L}}$-homomorphism.*

PROOF. The proof is a straightforward consequence of the definitions. □

4. Duality between $\mathcal{MMV}_n^{\mathcal{L}}$ and $\mathcal{MX}_n^{\mathcal{L}}$

The previous representation results can be left at a categorical level.

DEFINITION 5.14. We denote by $\mathcal{MX}_n^{\mathcal{L}}$ the category whose objects are the $\mathcal{MX}_n^{\mathcal{L}}$-structures and whose morphisms are the $\mathcal{MX}_n^{\mathcal{L}}$-morphisms. As usual, if X and Y are two objects of $\mathcal{MX}_n^{\mathcal{L}}$, we denote by $\mathcal{MX}_n^{\mathcal{L}}(X,Y)$ the set of the $\mathcal{MX}_n^{\mathcal{L}}$-morphisms from X to Y.

THEOREM 5.15 (Duality for $\mathcal{MMV}_n^{\mathcal{L}}$). *Let us denote by $\mathsf{D}_n^* : \mathcal{MMV}_n^{\mathcal{L}} \to \mathcal{MX}_n^{\mathcal{L}}$ the functor defined by*

$$\begin{cases} \mathsf{D}_n^*(A) = \langle \mathsf{D}_n(A), \{R_i \mid i \in I\}\rangle \text{ for any object } A \text{ of } \mathcal{MMV}_n^{\mathcal{L}}, \\ \mathsf{D}_n^*(f) : \mathsf{D}_n^*(B) \to \mathsf{D}_n^*(A) : u \mapsto u \circ f \text{ for any arrow } f \text{ in } \mathcal{MMV}_n^{\mathcal{L}}(A,B), \end{cases}$$

if R_i denotes the canonical k_i+1-ary modal relation associated to ∇_i for any i in I.

Let us also denote by $\mathsf{E}_n^* : \mathcal{MX}_n^{\mathcal{L}} \to \mathcal{MMV}_n^{\mathcal{L}}$ the functor defined by

$$\begin{cases} \mathsf{E}_n^*(X) = \langle \mathsf{E}_n(X), \{\nabla_i \mid i \in I\}\rangle \text{ for any object } X \text{ of } \mathcal{MX}_n^{\mathcal{L}} \\ \mathsf{E}_n^*(\psi) : \mathsf{E}_n^*(Y) \to \mathsf{E}_n^*(X) : \alpha \mapsto \alpha \circ \psi \text{ for any arrow } \psi \text{ in } \mathcal{MX}_n^{\mathcal{L}}(X,Y), \end{cases}$$

if ∇_i denotes the canonical k_i-ary dual MV-operator associated to R_i for any i in I.

Then, the functors D_n^* and E_n^* define a categorical duality between $\mathcal{MMV}_n^*$ and $\mathcal{MX}_n^*$.

PROOF. The easy details are left to the reader. □

First note that, by setting $n = 1$, one easily can easily realize that this duality is equivalent to the well known duality for boolean algebras with $\mathcal{L}$-operators. Hence, the duality is not a natural duality (in the sense of [**11**]).

We already know that if ∇ and ∇' are two k-ary dual MV-operators on an MV$_n$-algebra A such that $\nabla\!\restriction_{\mathfrak{B}(A)^k} = \nabla'\!\restriction_{\mathfrak{B}(A)^k}$ then ∇ and ∇' are equal. Nevertheless, we have not been able yet to provide a criterion that specifies the dual operators on $\mathfrak{B}(A)$ that can be extended to a dual MV-operator on A. We can now obtain such a criterion a a consequence of Theorem 5.15. Indeed, the dual of an operator of boolean algebra ∇' on $\mathfrak{B}(A)$ is a closed and continuous relation R' on the STONE dual of $\mathfrak{B}(A)$ (the underlying topological space of $\mathsf{D}_n(A)$) which in turns is or is not a modal relation on $\mathsf{D}_n(A)$.

PROPOSITION 5.16. *Assume that ∇' is a k-ary dual operator of boolean algebra on the algebra $\mathfrak{B}(A)$ of idempotent elements of an MV-algebra A. There is a k-ary dual MV-operator ∇ on A such that $\nabla\restriction_{\mathfrak{B}(A)^k} = \nabla'$ if and only if the dual $k+1$-ary relation R' associated to ∇' on $\mathsf{D}_n(\mathfrak{B}(A))$ is (up to the canonical homeomorphism of Lemma 5.4) a $k+1$-ary modal relation on $\mathsf{D}_n(A)$.*

Next, we prove that this duality provides us with a relational semantic which is complete with respect to any $Ł_n$-valued modal $\mathcal{L}$-logic.

DEFINITION 5.17. Assume that X is an $\mathcal{MX}_n^{\mathcal{L}}$-structure. A *valuation* on X is a map $\mathrm{Val} : X \times \mathsf{Prop} \to Ł_n$ such that $\mathrm{Val}(\cdot, p)$ belongs to $\mathcal{X}_n(X, \underset{\sim}{Ł}_n)$ for any propositional variable p of Prop.

Valuations are extended to formulas in the well known way and *models based on $\mathcal{MX}_n^{\mathcal{L}}$-structures* are defined in the natural way: such a model $\mathcal{M}$ is given by an $\mathcal{MX}_n^{\mathcal{L}}$-structure X together with a valuation on X.

Thus, if X is an $\mathcal{MX}_n^{\mathcal{L}}$-structure, a valuation Val on X is a valuation on its underlying $Ł_n$-valued $\mathcal{L}$-frame such that $\mathrm{Val}(\cdot, p) : X \to \underset{\sim}{Ł}_n$ is a continuous map for any propositional variable p of Prop.

PROPOSITION 5.18. *Assume that $\Theta \cup \{\phi\}$ is a subset of $\mathsf{Form}_{\mathcal{L}}$. The formula ϕ is a theorem of $\mathbf{K}_n + \Theta$ if and only if ϕ is valid in any $\mathcal{MX}_n^{\mathcal{L}}$-structure in which the formulas of Θ are valid.*

PROOF. The formula ϕ is a theorem of $\mathbf{K}_n + \Theta$ if and only if the equation associated to ϕ is valid in the variety of the $\mathbf{K}_n + \Theta$-algebras, or equivalently if ϕ is valid in any model based on the dual of a $\mathbf{K}_n + \Theta$-algebra. $\square$

Eventually, since the duality defined by D_n^* and E_n^* maps embeddings to onto morphisms and conversely, it is easy to realize that the lattice of congruences of an $\mathcal{MMV}_n$-algebra A i isomorphic to the order dual of the lattice of closed generated substructures of D^*A. But, since there is only one way to consider a closed hereditary subset of the underlying topological space of $\mathsf{D}^*(A)$ as a generated closed substructure of $\mathsf{D}^*(A)$, the lattice of closed generated substructures of $\mathsf{D}^*(A)$ is isomorphic to the lattice of the closed hereditary subsets of $\mathsf{D}_1(\mathfrak{B}(A))$. This line of argument is actually a proof of Proposition 3.17 that was announced in the third chapter.

5. Coproducts in $\mathcal{MX}_n^{\mathcal{L}}$

Coproducts of dual structures are classical constructions that one computes when one wants to obtain new members of the dual category. For example, the job has been done in **[35]** for the dual categories of boolean algebras with operators and has been considered in **[47]** for the members of $\mathcal{X}_n$. The problems in these constructions arise mainly from topology: when one computes non finite coproducts, one has to pay attention to preserve compacity and to conserve closed relations in order to stay in the category. The idea is to base the coproducts of the structures $(X_j)_{j\in J}$ on the STONE-ČECH compactification of the topological sum of the topological spaces X_j $(j \in J)$.

In fact, we can carefully merge the results of [**35**] and [**47**] to obtain the construction of the coproducts in $\mathcal{MX}_n$. The crucial point is to take care that the condition of R-saturation of the sets r_m is still satisfied in the compactification.

Let us recall the construction of the STONE-ČECH compactification of a completely regular topological space X. We denote by $C(X)$ the set of the continuous maps from X to $[0,1]$. Then, the evaluation map $e : X \to [0,1]^{C(X)}$ defined by $(e(x))_f = f(x)$ is a continuous map and is a homeomorphism from X to $e(X)$. If $\beta(X)$ denotes the closure of $e(X)$ in $[0,1]^{C(X)}$ then $(e, \beta(X))$ is the STONE-ČECH *compactification* of X. We set the notation $\bar{Y}$ aside to denote the closure in $\beta(X)$ of a subset Y of $\beta(X)$ and we identify X and $e(X)$ in $\beta(X)$. Finally, note that the coproduct of the boolean spaces X_j $(j \in J)$ in the category of boolean spaces with continuous maps is given by the STONE-ČECH compactification of the topological sum of the X_j. The set X_j can always be considered as being pairwise disjoint (otherwise we can replace X_j by $\{(x,j) \mid x \in X_j\}$ for all $j \in J$ with the obvious topology).

Note that the clopen subsets of $\beta(X)$ are exactly the $\bar{\Omega}$ where Ω is a clopen subset of X and that $\beta(X) \setminus \bar{F} = (X \setminus F)^-$ for every closed subspace F of X.

We provide the proofs of the following two lemmas even if they are part of folkore and can be found in [**35**], since the cited paper is not easily accessible.

LEMMA 5.19. *If X is a topological space whose set of clopen subsets is a base of the topology and if R is a closed $k+1$-ary relation on X, then $R^{-1}(K)$ is a closed subspace of X for every compact subspace K of X^k.*

PROOF. The proof is obtained thanks to a standard compacity argument. □

LEMMA 5.20. *Assume that J is a non empty set, that $(X_j)_{j\in J}$ is a family of boolean spaces and that $\beta(X)$ is the* STONE-ČECH *compactification of the topological sum X of the X_j.*

(1) *If F and F' are two disjoint closed subspaces of X, then $\bar{F}$ and $\bar{F'}$ are disjoint in $\beta(X)$.*
(2) *If F and F' are two closed subspaces of X, then $(F \cap F')^- = \bar{F} \cap \bar{F'}$ in $\beta(X)$.*
(3) *If R is a closed $k+1$-ary relation on X and if $\bar{R}$ denotes its closure in $\beta(X)$ then $\bar{R}^{-1}(\bar{\Omega}_1 \times \cdots \times \bar{\Omega}_k) = (R^{-1}(\Omega_1 \times \cdots \times \Omega_k))^-$ for every clopen subsets $\Omega_1, \cdots, \Omega_k$ of X.*

PROOF. (1) Let us denote by Ω_j a clopen subset of X_j such that $X_j \cap F \subseteq \Omega_j$ and $X_j \cap F' = \emptyset$ for all $j \in J$ and by Ω the open set $\bigcup\{\Omega_j \mid j \in J\}$. Thus, $\bar{F} \subseteq \bar{\Omega}$ and $\bar{F'} \subseteq (X \setminus \Omega)^- = X \setminus \bar{\Omega}$ since Ω_j is a zero-set in X_j for all $j \in J$.

(2) We prove the non trivial inclusion: let x be an element of $\bar{F} \cap \bar{F'}$ and Ω a clopen subset of X such that $x \in \bar{\Omega}$. We prove that $(\bar{\Omega} \cap F) \cap (\bar{\Omega} \cap F') \neq \emptyset$. Otherwise, it follows by (1) that $(\Omega \cap F) \cap (\Omega \cap F') = \emptyset$. But, since $\Omega \cap F$ is a closed subspace of X,

$$x \in \bar{\Omega} \cap \bar{F} \subseteq (\bar{\Omega} \cap F)^- = (\Omega \cap F)^-,$$

and we obtain similarly that $x \in (\Omega \cap F')^-$.

(3) The inclusion $(R^{-1}(\Omega_1 \times \cdots \times \Omega_k))^- \subseteq \bar{R}^{-1}(\bar{\Omega}_1 \times \cdots \times \bar{\Omega}_k)$ follows directly from Lemma 5.19. For the other inclusion, let x be an element of $\bar{R}^{-1}(\bar{\Omega}_1 \times \cdots \times \bar{\Omega}_k)$ and $\bar{U}$ be a clopen neighborhood of x in $\beta(X)$ (*i.e.*, U is a clopen subset of X). Then, there are $x_1 \in \bar{\Omega}_1$, $\ldots$, $x_k \in \bar{\Omega}_k$ such that $(x, x_1, \ldots, x_k)$ belongs to $\bar{R}$. Hence, the set $\bar{U} \times \bar{\Omega}_1 \times \cdots \times \bar{\Omega}_k$ is a

neighborhood of $(x, x_1, \ldots, x_k)$ in $\beta(X)$ and there is an x' in U, there are some x_1' in Ω_1, ..., x_k' in Ω_k such that

$$(x', x_1', \ldots, x_k') \in (U \times \Omega_1 \times \cdots \times \Omega_k) \cap R.$$

We eventually find that $R^{-1}(\Omega_1 \times \cdots \times \Omega_k) \cap U$ is not empty. $\square$

The idea to use STONE-ČECH compactification to compute coproducts in $\mathcal{X}_n$ can be traced back to [**47**]. We extend this construction to the category $\mathcal{MX}_n$.

PROPOSITION 5.21. *If J is an non empty set and*

$$X_j = \langle X_j, \{r_m^{X_j} \mid m \in \operatorname{div}(n)\}, \{R_i^{X_j} \mid i \in I\}\rangle$$

is an $\mathcal{MX}_n^{\mathcal{L}}$-structure for any j in J (we consider that the X_js are pairwise disjoint), then the structure

$$\langle \beta(X), \{(r_m^X)^- : m \in \operatorname{div}(n)\}, \{\overline{R}_i \mid i \in I\}, \tau\rangle$$

where

- *X is the topological sum of the X_j ($j \in J$),*
- *$\beta(X)$ is the* STONE-ČECH *compactification of X and X is identified with $e(X)$ in $\beta(X)$,*
- *r_m^X is the the subspace $\bigcup\{r_m^{X_j} \mid j \in J\}$ for every $m \in \operatorname{div}(n)$*
- *R_i is the relation $\bigcup\{R_i^{X_j} \mid j \in J\}$ for any i in I,*

is the coproduct in $\mathcal{MX}_n^{\mathcal{L}}$ of the X_j where $j \in J$.

PROOF. We first have to prove that the proposed structure is an object of $\mathcal{MX}_n$. First, it is clear that its underlying topological space is a boolean space. The identities

$$r_m^{\beta(X)} \cap r_{m'}^{\beta(X)} = r_{gcd(m,m')}^{\beta(X)}$$

where m and m' are two divisors of n are obtained as a consequence of item (2) of Lemma 5.20.

We now prove that $\bar{R}$ is a modal relation on $\beta(X)$ if R is a $k+1$-ary relation of the language. First of all, the third item of Lemma 5.20 implies that $\bar{R}^{-1}(\Omega_1 \times \cdots \times \Omega_k)$ is clopen subset for every clopen subsets Ω_1, ..., Ω_k of $\beta(X)$. To prove that

$$\bar{R}(r_m^{\beta(X)}) \subseteq r_m^{\beta(X)^k},$$

we proceed *ad absurdum*. Assume that $(x, x_1, \ldots, x_k) \in \bar{R}$ with $x \in r_m^{\beta(X)}$ but $x_l \in \beta(X) \backslash r_m^{\beta(X)}$ for an l in $\{1, \ldots, k\}$. Let us consider a clopen subset Ω of X such that

$$y \in \bar{\Omega} \subseteq \beta(X) \setminus r_m^{\beta(X)}.$$

Then, thanks to item (3) of Lemma 5.20, we obtain that x belongs to $\bar{R}_{(l)}^{-1}(\bar{\Omega}) = \overline{R_{(l)}^{-1}(\Omega)}$. We thus can find a $t \in r_m^X \cap R_l^{-1}(\Omega)^- = r_m^X \cap R^{-1}(\Omega)$. Finally, it means that there is a $z \in \Omega$ and $z_1, \ldots, z_{l-1}, z_{l+1}, \ldots, z_k$ in X such that

$$(t, z_1, \ldots, z_{l-1}, z, z_{z+1}, \ldots, z_k) \in R,$$

which is a contradiction since $\Omega \subseteq X \setminus r_m^X$. We so have proved that the subspaces $r_m^{\beta X}$ satisfy the condition of $\bar{R}$-saturation of Definition 5.7, and have finished to prove that the proposed structure belongs to $\mathcal{MX}_n^{\mathcal{L}}$.

Now, let us prove that we have computed the coproduct of the X_i. We denote by $\sigma_i : X_i \to X$ the inclusion map of X_i into $\beta(X)$ for every $i \in I$. These maps are obviously $\mathcal{MX}_n^{\mathcal{L}}$-morphisms (use the fact that $\bar{R}(u) = R(u)$ for every $u \in X$).

Then, suppose that $f_i : X_i \to Y$ is an $\mathcal{MX}_n^{\mathcal{L}}$-morphism which is valued in an $\mathcal{MX}_n^{\mathcal{L}}$-structure Y for every $i \in I$. Since $\beta(X)$ is the coproduct of the topological spaces X_i, there is a unique continuous map $f : \beta(X) \to Y$ such that $f \circ \sigma_i = f_i$ for every $i \in I$. We prove that

$$f((r_m^X)^-) \subseteq r_m^Y$$

for every divisor m of n. First, assume that $y \in Y \setminus r_m^Y$ and denote by Ω a clopen subset of Y such that $y \in \Omega$ and $\Omega \cap r_m^Y = \emptyset$. It follows that $f^{-1}(y) \subseteq f^{-1}(\Omega) \subseteq \beta(X) \setminus r_m^X$. Thus, $f^{-1}(y) \subseteq \beta(X) \setminus (r_m^X)^-$ which proves that f is an $\mathcal{X}_n$-morphism.

Finally, we prove that f is a bounded morphism, *i.e.*, that

$$f(\bar{R}(u)) = R^Y(f(u))$$

for every u in $\beta(X)$. The inclusion from left to right is easily obtained. We proceed with the other inclusion. Assume that $x \in R^Y(f(u))$. It suffices to show that every clopen neighborhood V of x meets $f(\bar{R}(u))$ since the latter is closed. Let Ω be any clopen subset of X such that $\bar{\Omega}$ contains u. It then follows that

$$R^Y(f(\bar{\Omega})) = R^Y((f(\Omega))^-) \subseteq (R^Y(f(\Omega)))^- = (f(R^X(\Omega)))^- = f((R(\Omega))^-) = f(\bar{R}(\bar{\Omega})).$$

Hence, since $x \in R^Y(f(u)) \subseteq f(\bar{R}(\bar{\Omega}))$, the intersection $\bar{R}^{-1}(f^{-1}(V)) \cap \bar{\Omega}$ is not empty. We obtain that u belongs to $\bar{R}^{-1}(f^{-1}(V))$ since this subspace is closed in $\beta(X)$. It means that V contains an element of $f(\bar{R}(u))$ and so that $x \in (f(\bar{R}(u)))^- = f(\bar{R}(u))$. $\square$

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