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Transient Fokker-Planck Equation solved with SPH

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Abstract

In many engineering matters, systems are submitted to random excitations. Probabilistic theories aim at describing the properties of a system by means of statistical properties such as probability density function ψ (pdf). For a deterministic system randomly excited, the evolution of its pdf is commonly described with Fokker-Planck-Kolmogorov equation (FPK). The FPK equation is a conservation equation of a hypothetical fluid, which represents physically the transport of probability in $\mathbb{R}^n$. To solve this equation, Smoothed Particle Hydrodynamics (SPH) are used: the system is modelled with a conservation equation for the system and a transport equation for each particle.

Numerical implementation shows the superiority of this method over many other mesh-based methods: (i) the conservation of total probability in the state space is explicitly written, (ii) no specific boundary conditions must be imposed if an adaptive smoothing length is chosen and if particles are initially regularly spread out, (iii) the positivity of the pdf is ensured.

Furthermore, thanks to the moving particles, this method is adapted for a large kind of initial conditions (*quasi*-deterministic or even discontinuous). The FPK equation can be solved without any *a priori* knowledge of the stationary distribution; just a precise representation of the initial distribution is required.

1 Introduction

In many engineering matters, systems are submitted to random excitations. Probabilistic theories aim at describing the time evolution of properties of a system by means of statistical characteristics such as probability density function ψ (pdf) or cumulants. For an assumed deterministic system excited by random loadings, if the time scale between the system and the excitations are clearly different [1], the time evolution of the pdf is given by the Fokker-Planck-Kolmogorov equation (FPK).

Let us focus on deterministic non-linear systems excited by random inputs. The time evolution of the random state vector $\mathbf{X}(t) \in \mathbb{R}^n$ is mathematically described by a stochastic differential equation [1]

$$d\mathbf{X} = \mathbf{f}dt + \mathbf{g}\,\mathbf{W}dt \tag{1}$$

where $\mathbf{f}(t, \mathbf{X}(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector function, $\mathbf{g}(t, \mathbf{X}(t)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ a nonlinear matrix function and $\mathbf{W}(t)$ a m-dimensional white noise vector. The FPK equation expresses the diffusion of probability density of $\mathbf{X}(t)$ and the conservation of the total probability. It takes the form of a second order partial differential equation:

$$\frac{\partial \psi}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\tilde{f}_i \psi\right) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left(D_{ij}\psi\right) \tag{2}$$

where $\psi(\mathbf{X}(t), t) : \mathbb{R}^n \times \mathbb{R} \to [0, +\infty]$ is the joint-pdf of elements of $\mathbf{X}(t)$ with the Kolmogorov properties

$$\int_{\mathbb{R}^n} \psi d\mathbf{x} = 1 \qquad \text{and} \qquad \psi(\mathbf{x}, t) > 0, \forall \mathbf{x} \in \mathbb{R}^n. \tag{3}$$

and where $\widetilde{f}_i(t, \mathbf{X}(t))$ are the drift coefficients (according to Stratonovitch's definition) and $D_{ij}(t, \mathbf{X}(t))$ the diffusion coefficients defined as $D_{ij} = 1/2(\mathbf{g}\mathbf{g}^T)_{ij}$. The distinction between $\tilde{f}_i(t, \mathbf{X}(t))$ and $f_i(t, \mathbf{X}(t))$, due to the conceptual definition of the stochastic differential equations, is noteworthy in the event of a system excited by a random parametric excitation (see [2]).

For a linear deterministic system subjected to a Gaussian random excitation, the response is also Gaussian. The time evolution of the pdf is perfectly characterized by the time evolution of the first- and second-order cumulants. For a nonlinear dynamical system, non-Gaussianity complicates the solution of the FPK equation and the use of numerical methods is necessary.

The first numerical technique used to solve FPK equation was the finite difference method [3]. In further attempts, the finite element method has also been used [4]; it has recently been extended to a four-dimensional state space. This method allows to solve the non stationary equation through a step-by-step analysis or immediately the stationary equation through an eigenvalue problem. Nevertheless, the finite element method encounters difficulties in the boundary regions. To overcome this problem, Spencer [4] proposes to mesh a sufficiently large domain to eliminate the diffusion of probability. Langtangen [5] proposes to impose a zero flux condition at the boundaries of the integration domain. However, the positivity of the pdf is not properly ensured: if elements are too large or the domain is too small some spurious waves can propagate through the state space and spoil the quality of the solution. The finite element method is extended with difficulties to multi-dimensional systems in spite of some recent improvements though. An argument against mesh-based methods is the consideration of useless subspaces: at a certain time step, probability density in some subspaces is so low that its representation is useless. For a given computing capacity, it is more useful to refine appropriate zones of the state space.

In this work, the FPK equation viewed as a diffusion-convection equation in a Lagrangian form, represents physically the transport of a probability density across a state space. This particular equation is similar to governing equations encountered in heat conduction or fluids dynamics problems. In these fields, Smoothed Particle Hydrodynamics (SPH) method, is often used [6, 7, 8]. In this famous meshless method, each particle carries an invariable mass across a given space. The evolution of the probability density of a particle depends on the volume taken by this particle, depending on the proximity and the motion of the others particles.

2 Lagrangian formulation of FPK equation

In the context of fluid dynamics, an Eulerian description of motion can be imagined as that given by a fixed observer looking at the evolution of fluid properties in a finite, fixed and undeformable volume. Contrarily, a Lagrangian description of motion can be viewed as the motion described by an observer sitting on a moving particle. These views modify the formulation and the interpretation of the equations of motion: in a Lagrangian formalism, the integration points are moving according to trajectories depending on the modeled system.

The Eulerian formulation of a convection-diffusion phenomenon of a scalar field ϕ is related to its Lagrangian formulation through the concept of *total derivative* D/Dt,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot (\nabla \phi) = \frac{\partial \phi}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial \phi}{\partial x_i} \tag{4}$$

where $\mathbf{v}$ is the medium velocity.

For the FPK equation (2), such a velocity field $\mathbf{v}(\mathbf{X}, t)$ is introduced

$$\frac{\partial \psi}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\psi \left(\tilde{f}_i - \sum_{j=1}^{n} \frac{\partial D_{ij}}{\partial x_j} - \frac{1}{\psi} \sum_{j=1}^{n} D_{ij} \frac{\partial \psi}{\partial x_j} \right) \right) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} (v_i \psi) \tag{5}$$

where $v_i(\mathbf{X}, t)$ is the component i of $\mathbf{v}(\mathbf{X}, t)$ given by

$$v_i = \tilde{f}_i - \sum_{j=1}^{n} \frac{\partial D_{ij}}{\partial x_j} - \frac{1}{\psi} \sum_{j=1}^{n} D_{ij} \frac{\partial \psi}{\partial x_j} \,. \tag{6}$$

The velocity $\mathbf{v}$, so-called *diffusion velocity* [9], has a term divided by ψ, but there is no hidden difficulties. Indeed, if ψ represents a density in a wide sense, a particle cannot have a density equal to zero because it cannot occupy an infinite volume. For a dynamical time-invariant system without parametric excitation, (6) becomes

$$v_i = f_i - \frac{1}{\psi} \sum_{j=1}^{n} D_{ij} \frac{\partial \psi}{\partial x_j} \,. \tag{7}$$

The conservation equation (5) is further developed to introduce the total derivative

$$\frac{\partial \psi}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (v_i \psi) = \frac{D\psi}{Dt} + \sum_{i=1}^{n} \psi \frac{\partial v_i}{\partial x_i} = \frac{D\psi}{Dt} + \psi \nabla \cdot \mathbf{v} = 0 \,. \tag{8}$$

and the vectorial equation of transport across the state space reads simply

$$\frac{d\mathbf{X}}{dt} = \mathbf{v} \tag{9}$$

with $\mathbf{v}$ the velocity field defined in (6).

In the following Section, the Smoothed Particle Hydrodynamics method is presented. This method is used to solve the FPK equation expressed in a Lagrangian formalism. Physically, SPH method models the transport equation (9) as the motion of particles across the state space.

3 Smoothed particle hydrodynamics

The philosophy of SPH method is to transform a set of partial differential equations into integral equations by using an interpolation function that gives the kernel estimation of a field (density, velocity, energy) at a point. Vector or scalar fields are only known in a number of discrete points. Thus, the evaluation of an integral is transformed to a sum over some neighbouring particles. In this method, a grid is unnecessary, because interactions between neighbouring particles are modelled with interpolation functions, so-called *kernel functions*, which depend on the distance between particles.

3.1 Integral representation of a field

The concept of *integral representation* of a field $f(\mathbf{x})$ in SPH method uses the properties of the Dirac-delta function $\delta(\mathbf{x})$, i.e.

$$\int_{\mathbb{R}^n} f(\mathbf{x}') \delta \left(\mathbf{x} - \mathbf{x}' \right) d\mathbf{x}' = f(\mathbf{x}) \tag{10}$$

with properties

$$\delta(\mathbf{x} - \mathbf{x}') = 0, \text{ if } \mathbf{x} \neq \mathbf{x}' \qquad \text{and} \qquad \int_{\mathbb{R}^N} \delta \left(\mathbf{x} - \mathbf{x}' \right) d\mathbf{x}' = 1 \,. \tag{11}$$

This function can not be numerically represented. Therefore, the Dirac delta function is replaced by a function $W(r,h)$, the *kernel function*, such that,

$$\lim_{h\to 0} W\left(|\mathbf{x}-\mathbf{x}'|,h\right) = \delta(\mathbf{x}-\mathbf{x}') \tag{12}$$

with h the *smoothing length* of $W(r,h)$. The *kernel approximation* of $f(\mathbf{x})$ with the kernel function is

$$< f(\mathbf{x}) >= \int_{\mathbb{R}^n} f(\mathbf{x}')W(|\mathbf{x}-\mathbf{x}'|,h)d\mathbf{x}' \,. \tag{13}$$

According to the *Delta function property* (12), $< f(\mathbf{x}) > \to f(\mathbf{x})$ when $h \to 0$. This last expression is a smoothed approximation of $f(\mathbf{x})$ and the integral is discretized, for a number N_p of particles, as

$$< f(\mathbf{x}) > \quad \approx \quad \sum_{j=1}^{N_p} f(\mathbf{x}_j)W(|\mathbf{x}-\mathbf{x}_j|,h)\Delta V_j = \sum_{j=1}^{N_p} \frac{m_j}{\rho_j} f(\mathbf{x}_j)W(|\mathbf{x}-\mathbf{x}_j|,h) \tag{14}$$

where m_j, ρ_j and ΔV_j are respectively the mass, the density and the volume of particle j. Equation (14) is the *particle approximation* of the field $f(\mathbf{x})$. This approximation can be particularized at a particle i,

$$< f(\mathbf{x}_i) >= \sum_{j=1}^{N_p} f(\mathbf{x}_j)W_{ij}\Delta V_j \tag{15}$$

with $W_{ij} = W(r_{ij}, h_i)$ and r_{ij} the distance between $\mathbf{x}_i$ and $\mathbf{x}_j$. Equation (15) highlights the difference between the particle value $f(\mathbf{x}_i)$ and the kernel approximation $< f(\mathbf{x}_i) >$. These values must not be confused but can be sufficiently close in some cases.

3.2 Kernel function

A kernel function must fulfil some conditions to be used in SPH method:

- *Unity*: the kernel function must be normalized

$$\int_{\mathbb{R}^n} W(|\mathbf{x}-\mathbf{x}'|,h)d\mathbf{x}' = 1 \qquad \text{or} \qquad \sum_{j=1}^{N} W(|\mathbf{x}-\mathbf{x}_j|,h)\Delta V_j = 1 \tag{16}$$

- *Delta function property*: this condition imposes that when h tends to zero, the kernel function tends to a Dirac function (12).
- *Compact support:* the kernel function should be compactly supported: $W(r,h) = 0$ if $r \geq \kappa h$, where κ is a constant depending on the kernel function.
- *Decay*: the smoothing function should decrease monotonically with the distance between a particle and the neighbouring ones.
- *Symmetry , positivity and smoothness*.

In this work, the original Lucy kernel function is used. Introduced by Lucy in 1977 [10], this kernel is a polynomial function continuously derivable over its compact support ($\kappa = 1$),

$$W(R,h) = \alpha_d \begin{cases} (1+3R)\,(1-R)^3, & \text{if } R \leq 1 \\ 0, & \text{if } R > 1 \end{cases} \tag{17}$$

with $R = r/h$. The parameter α_d depends on the dimension n of the state space. From (17), α_d is $5/4h$, $5/\pi h^2$ and $105/16\pi h^3$ in one-, two- and three-dimensional spaces.

3.3 Smoothing length

The selection of the smoothing length h is an important issue in the SPH method. If the smoothing length is too small, a particle does not interact with its neighbouring particles. On the other hand, if the smoothing length is too large, local properties of the studied field could be inaccurately smoothed. One commonly assumes that the accuracy depends on h^2.

The smoothing length also influences the computation time. For these reasons, the smoothing length associated to each particle is adapted at each time step. Liu [7] suggests to maintain an appropriate number of particles in the neighbourhood of each particle: 5, 21 and 57 in respectively one-, two- and three-dimensions.

To adapt the smoothing length with particle position, different authors suggest relations to maintain the number of neighbouring particles constant. The simplest relation consists in keeping the product $(h_0 \sqrt[d]{\rho_0})_i$ for each particle constant in time, where d is the dimension of the considered space. The smoothing length is updated with the evolution of ρ_i, thus

$$h_i = h_{0,i} \left(\sqrt[d]{\frac{\rho_0}{\rho}} \right)_i . \tag{18}$$

Interaction between particles is a key problem in SPH method. Indeed, a particle i interacts only with particles contained in its compact support. Therefore, it is useless to check possible interaction if two particles are too far from each other. Specific literature widely covers this topic [6, 7].

4 SPH formulation of Fokker-Planck equation

4.1 Conservation equation

The *amount of probability* μ_j carried by a particle j, analogous to mass in fluid, is defined as the product of the particle volume ΔV_j and the particle density ψ_j. Taking into account (15), if the field of interest is the density of probability, the particle approximation of ψ at particle i is [8],

$$< \psi(\mathbf{x}_i) >= \sum_{j=1}^{N_p} \psi_j W_{ij} \Delta V_j = \sum_{j=1}^{N_p} \mu_j W_{ij} \tag{19}$$

where $\mu_j = \psi_j \Delta V_j$. The mass of each particle is surely conserved and so is the total probability. The conservation of mass is implicitly formulated in the particle approximation depending on particle positions. Therefore, from a numerical point of view, there is no need to solve the conservation equation as a differential equation. At this stage, the positivity of the pdf is ensured, because each particle approximation is a sum of positive terms.

With this formulation the mass is conserved, but the sum of masses is not exactly equal to one. Actually, masses are initially calculated according to the initial probability density function $\mu_j = \psi_{0,j} \Delta V_j$. (a regular initial mesh is appropriated for this approximation) To keep the total mass constant, it is important to have $W_{ij} = W_{ji}$. The chosen method consists in averaging the kernel functions between two particles, i.e.

$$W_{ij} = W_{ji} = \frac{1}{2} \left(W(r_{ij}, h_i) + W(r_{ij}, h_j) \right) . \tag{20}$$

4.2 Transport equation

According to (19), the probability density of a particle i depends on W_{ij} and therefore on particle positions at time t. To calculate the position of a particle i, (9) is integrated. A simple Eulerian integration scheme is sufficient for such a simple differential equation

$$\mathbf{X}_i^{t+\Delta t} = \mathbf{X}_i^t + \mathbf{v}_i^t \Delta t \tag{21}$$

The velocity is discretized, according to the concept of integral approximation of a vector field. Equation (6) becomes

$$\mathbf{v}(\mathbf{X}) \quad = \quad \mathbf{f}(\mathbf{X}) - \mathbf{D}\frac{\nabla\psi(\mathbf{X})}{\psi(\mathbf{X})} \tag{22}$$

and the particle approximation of (22) is

$$\mathbf{v}_i^t = \mathbf{f}(\mathbf{X}_i^t) - \mathbf{D}\frac{<\nabla\psi(\mathbf{X}_i^t)>}{<\psi(\mathbf{X}_i^t)>} \tag{23}$$

where $<\psi(\mathbf{X}_i^t)>$ is calculated with (19) and $<\nabla\psi(\mathbf{X}_i^t)>$ calculated as

$$<\nabla\psi(\mathbf{X}_i^t)>=\sum_{j=1}^{N_p}\mu_j\nabla_i W_{ij}\,. \tag{24}$$

The Lagrangian formalism makes the method relatively independent with regard to the final solution. With SPH, particles move from the initial to the stationary distribution by themselves. Contrarily, in mesh-based method, the mesh must cover a large space to be able to represent every transient step of the solution.

4.3 Some computational aspects

Particle Interaction

Through this discretization, a single equation must be solved at the beginning of the computation: knowing initial particle positions and initial density field in each particles, (19) is solved to calculate masses. However, masses can be reasonably well calculated by the relation $\mu_j = \psi_j \Delta V_j$. Hence, in any transient simulation, there is no equation to solve. Particle motion modifies the volume occupied by each particle and therefore the associated density. If particles are too close, their densities increase and their smoothing lengths decrease. Conversely, if a particle is distant from the others, its density decreases and its smoothing length increases. Therefore, a particle cannot be isolated without any interaction with its neighbours. There is no risk that a particle clears off.

To compute interaction, particles are first sorted in cells. This mapping consists in dividing the state space in a number of cells. In some applications, the dimension of hypercubes is chosen as the maximum smoothing length h over all particles. This choice is adequate only if the density in the system is quite homogeneous. In the case of probability density evolution, there are heavy differences between zones of the state space. Therefore, the mapping is constructed with a given number of cells N_c and the dimensions of the hypercubes is modified at each time step.

Time step

If particles become too close, their probability density can explode. Generally, this phenomenon is avoided if the time step is properly adapted. The time step is a key factor of stability in this method. Indeed, some sets of particles initially separated can meet at a certain time. New interactions occur and must be computed. A time step adapted to the particle velocities allows a smooth mixing and a smooth evolution of the interaction, because the distance between particles is smoothly reduced.

In this work, the time step is calculated as

$$\Delta t = C_r \min_{i\in[1,N_p]}\left(\frac{r_{min,i}}{v_i}\right), \quad \text{and} \quad r_{min,i} = \min_{1\leq j\leq N_p, i\neq j}(r_{ij}) \tag{25}$$

where C_r is the Courant number.

In [7], the time step is chosen as the minimum ratio h_i/v_i over all particles. Here, the characteristic distance in the Courant number is the minimum distance between a particle and its neighbours $r_{min,i}$. The physical meaning of this choice is that a particle cannot cover a distance greater than a fraction (given by the Courant number) of this minimum distance in a time step. This choice offers a sufficient, but not necessary condition to stability.

5 Results

5.1 Two-dimensional random system

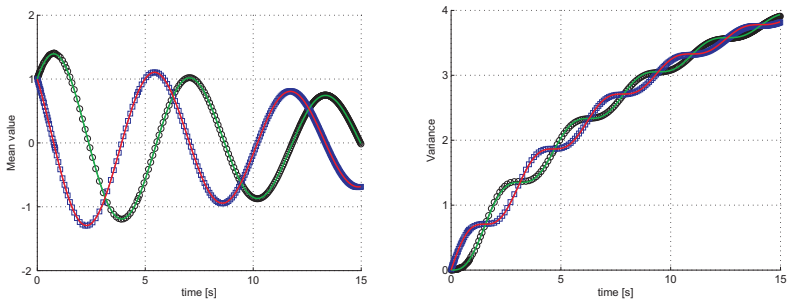
Linear oscillator

Let us consider a 1-DOF linear oscillator

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = W \tag{26}$$

where $W(t)$ is a Gaussian δ-correlated noise with zero mean and $E[W(t)W(t+\tau)] = 2D\delta(\tau)$. For this linear oscillator, the analytical expressions of time evolutions of means and variances, starting from deterministic initial conditions, are well known [11]. For the system, the parameters are $\omega_0 = 1$, $D = 1$ and $\xi = 0.05$ or $\xi = 0.5$. Initial conditions are $\mu_x = \mu_{\dot{x}} = 1$, $\sigma_x = \sigma_{\dot{x}} = 0.1$ and $\rho_{x\dot{x}} = 0$, to have *quasi*-deterministic initial conditions.

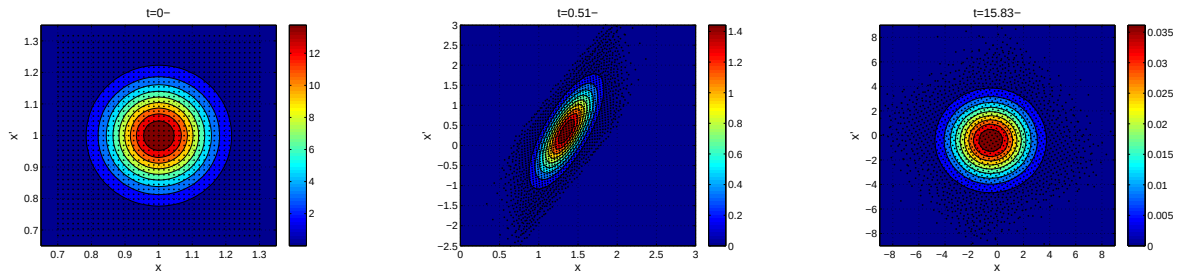
Figure 1 shows the time evolution of means and variances of x and $\dot{x}$. The damping ratio ξ is 0.05 to exhibit oscillations in the transient statistical properties of the system. The variances and the mean values match very well the analytical solution.



(a) Means (b) Variances

Figure 1: Linear oscillator (ξ =0.05%). Time evolution of the mean values and variances of x (□) and $\dot{x}$ (O) with 1681 particles. Solid lines are the analytical solutions.

Figure 2 presents three sketches of the pdf evolution[1]. The particles automatically move from the initially meshed zone to the subspace of interest at any time step.



(a) (b) (c)

Figure 2: Linear oscillator: $\psi(x, \dot{x})$ for three time steps. 1681 particles.

The capability of the program to reach the stationary solution is also checked on the same example with ξ equal to 0.5. The transient phase is therefore shorter. The FPK equation is solved for different numbers of particles N_p and the relative error on the pdf is calculated as

[1]The *griddata* function in Matlab is used to build the pdf between particles

$$\mathcal{E}_\psi = \sqrt{\sum_{i=1}^{N_p} |\psi - \psi_{ana}|_i^2 \Big/ \sum_{i=1}^{N_p} |\psi_{ana}|_i^2} \tag{27}$$

Table 1 gathers errors on the density and indicators of performance: error $\mathcal{E}_\psi$, the time reached after 2000 simulations, the CPU time for the simulation, the number of cells of the mapping N_c. The Courant number C_r is fixed to 0.2. In this example, the error on the stationary solution is not notably modified by an increment of the number of particles.

N_p	121	441	961	1681	2601	3721	3721	6561
$\mathcal{E}_\psi$	7.69%	0.95%	0.82%	1.27%	1.08%	1.02%	1.02%	1.31%
t_{end}	18.30s	18.87s	16.52s	14.02s	12.34s	12.43s	12.43s	9.90s
CPU	7.8s	18.2s	39.4s	72.9s	139.3s	???	220.8s	544.3s
N_c	100	100	100	400	900	100	900	900

Table 1: Linear oscillator (ξ =0.5%). 2000 time steps are computed for different number of particles N_p. Relative error $\mathcal{E}_\psi$, final time t_{end}, indicator CPU and the number of mapping cells N_c

Table 1 shows also that the adaptive time step depends on the number of particle: starting from the same initial condition, more particles implies closer particles. So a smaller time step must be chosen to correctly control the displacements and the interaction between particles. After 2000 steps, the less the number of particles, the longer the simulated time interval. In a Lagrangian formalism, the steady state is characterized by the motion of particles along lines of isoprobability. This is a main difference between Eulerian and Lagrangian formalisms.

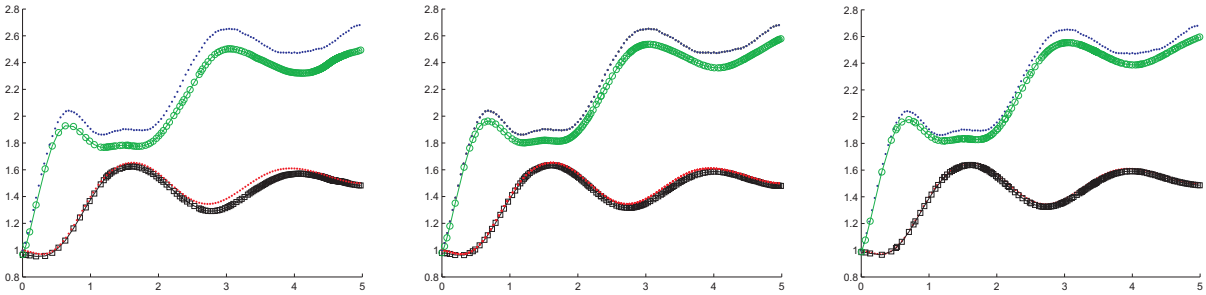
The mapping cells aims at finding quickly the neighbouring particles of a given particle. A too large number of cells is not necessary, because a lot of cells will be empty and useless. An adequate number of cells is chosen to improve the computation of the kernel functions W_{ij}. The CPU time may be dramatically increased if there are not enough cells.

Duffing-Van der Pol Oscillator

Secondly, a Duffing-Van der Pol oscillator excited by a white noise is considered,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + \omega_0^2(-x + x^3) = W(t) \tag{28}$$

with $\epsilon = 0.8$ and $\omega_0 = 1$. The initial condition is a standard Gaussian distribution characterized by $\mu_x = \mu_{\dot{x}} = 0$, $\sigma_x = \sigma_{\dot{x}} = \sqrt{2}/2$ and $\rho_{x\dot{x}} = 0$.

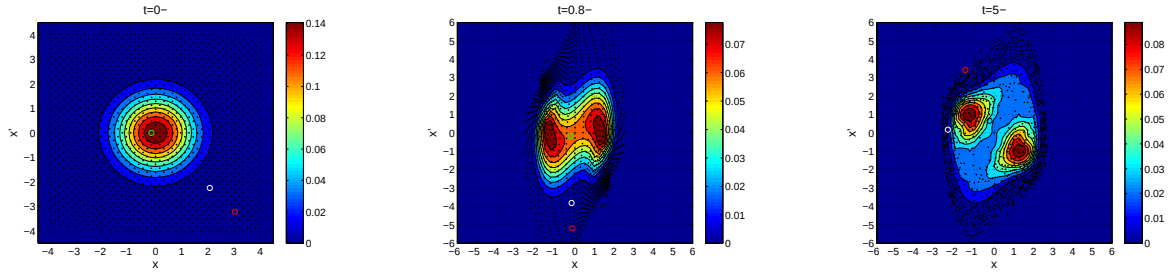


(a) 961 particles (b) 1681 particles (c) 2601 particles

Figure 3: Duffing-Van der Pol oscillator. Time evolution of the second-order moment of x (□) and $\dot{x}$ (O) for different number of particles. The dot lines are the results of Monte Carlo simulations

To evaluate the method in the transient phase, the system is computed until 5s with 961, 1681 and 2601 particles. The proposed method is compared with Monte-Carlo simulations in Fig. 3 (75000

samples). The accuracy of proposed method increases with the number of particles, but not notably. Three snapshots of the joint-pdf are shown in Fig. 4. Starting from a regular distribution, particles move and produce the expected pdf. These pictures show that particles are concentrated around the limit cycle and especially in the transition between fast and slow dynamics.



(a) (b) (c)

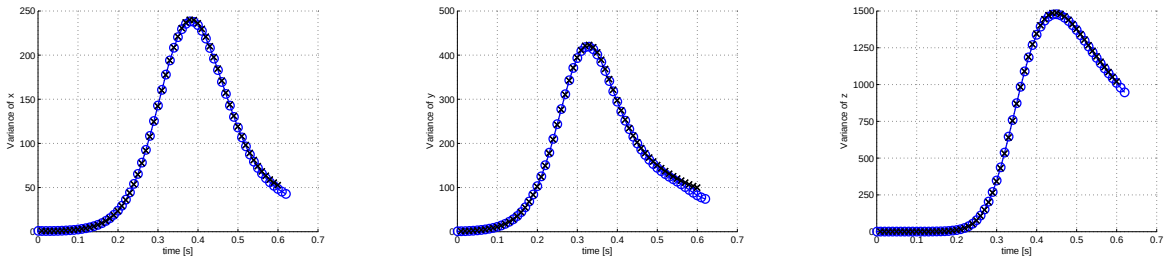
Figure 4: Duffing-Van der Pol oscillator: $\psi(x, \dot{x})$ for three time steps. 2601 particles.

5.2 Three-dimensional random system

For this last example, the Lorenz attractor, a three-dimensional non-linear oscillator, is studied:

$$\begin{aligned} \dot{x} &= \sigma(y - x) + W_1 \\ \dot{y} &= \rho x - y - xz + W_2 \\ \dot{z} &= xy - \beta z + W_3 \end{aligned} \tag{29}$$

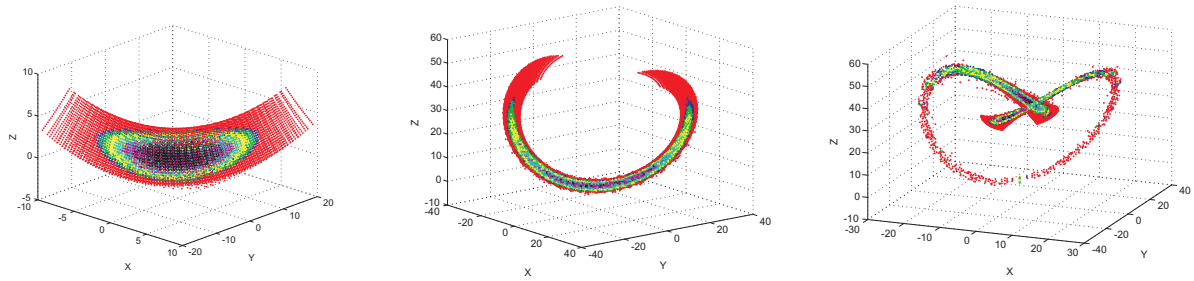
with $\sigma = 10$, $\rho = 28$, $\beta = 8/3$ and W_1, W_2, W_3 three independent unit Gaussian white noises. Fig. 5 shows the time evolutions of the second-order moments of the three state variables. The comparison with Monte-Carlo simulations highlights the efficiency of the method for a reasonable number of particles. With $N_c = 800$, the computation time is around 0.8s per time step for the simple proposed method.



(a) (b) (c)

Figure 5: Time evolutions of second-order moments of x (a), y (b), and z (c) computed with SPH (O) compared with Monte-Carlo results ($\times$) (75000 simulations). 9261 particles are considered

Fig. 6 illustrates positions of the particles in the three-dimensional state space at three time steps. The use of different colours is a didactic way to represent the density associated with each particle. Starting from a Gaussian initial condition, particles move in the three-dimensional space to finally produce the Lorenz attractor.



(a) $t = 0.10$ (b) $t = 0.25$ (c) $t = 0.50$

Figure 6: Lorenz oscillator: $\psi(x, y, z)$ for three time steps. 9261 particles are considered

6 Conclusion

In this paper, a Lagrangian method to solve FPK equation using SPH method has been exposed. Some results in two and three dimensions are shown to illustrate the accuracy and the main parameters of the method. To conclude, some advantages are summarized.

The Lagrangian formalism makes the method relatively independent with regard to the final solution. With SPH, particles move from the initial to the stationary distribution by themselves. Therefore, only an accurate representation of the initial condition (even *quasi*-deterministic) must be worried about. Furthermore, the SPH method ensures the positivity of the pdf.

Nevertheless, the method has also some limitations. For instance, the stationary distribution cannot be directly computed, contrary to finite element methods. In the context of extreme value problems, this method is limited because low density zones are not necessarily accurately represented. From a computational point of view, for a large number of particles, the computation of the interaction can turn out to be more time consuming than efficient algorithms.

Some improvements will be naturally realized in future researches. However, SPH method, as presented in this paper, offers already the possibility to quickly and easily extend the formalism to multidimensional spaces and to have an accurate representation of the transient phase of FPK equation.

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