Model based, gain-scheduled anti-windup control for LPV systems

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***Abstract*— In this paper we show that a recently proposed technique for anti-windup control of exponentially unstable plants can be easily extended to solve the corresponding robust anti-windup problem for linear parameter varying systems, for which the time varying parameters are measured online. For this class of plants, it is shown that the proposed technique is minimally conservative with respect to the size of the resulting operating region: in particular, such a region is (up to an arbitrarily small quantity) exactly the largest set on which asymptotic stability can be guaranteed for the considered plant, for the given saturation level and uncertainty characteristics.**

I. INTRODUCTION

Input saturation and plant uncertainty are two ubiquitous phenomena that a control engineer has to face in any real design problem. The presence of saturation nonlinearities in otherwise linear closed loop systems can cause dramatic performance losses known as "windup" effects; in order to avoid such losses, several anti-windup compensation techniques have been developed, starting with the pioneering, heuristic solutions proposed in the 1950's (*e.g.* [20]; see also the surveys [19], [1]) until general results with formal proofs of stability started appearing during the last decade (see *e.g.* [28], [21], [10], [11] and references therein). On the other hand, plant uncertainty motivated a huge amount of research in the field of robust control, where, under different assumptions about the uncertainty model (*e.g.* structured or unstructured, fixed or time-varying unknown parameters and so on), many different solutions have been proposed. A difference that will be of interest in this paper is the difference between *robustness-in-the-large* and *robustness-in-the-small* of a given property (*e.g.*, asymptotic stability of a closed loop system), where the property is said to be robust-in-the-small if it holds for any sufficiently small uncertainty, whereas it is said to be robust-in-the-large if it holds for any uncertainty in an *a priori* assigned (and perhaps "large") set of uncertainties. Usually, when a stabilization problem is solved in the nominal parameters, then robustness-in-the-small can be obtained (almost) "for free" by invoking suitable (possibly nonlinear) small gain theorems; this is not the case when robustness-in-the-large is of interest.

Taking into account the above discussion, it is somewhat remarkable that, as pointed out in [25], the problem of designing anti-windup compensators devoted to the study of robustness limitations specifically arising in anti-windup control systems, or that guarantee robust-in-the-large stability has not been systematically addressed in the literature.

In particular, even in the nominal parameters, the problem of anti-windup compensation is known to be especially challenging in the case on exponentially unstable plants, for which (due to the presence of bounds on the input) the null controllable region is bounded (see, *e.g.*, [22]) and then in order to achieve stability the anti-windup compensator has to take care that the state of the plant never leaves a suitable subset of the null controllable region. Anti-windup designs for exponentially unstable linear plants have been recently suggested in a number of papers, including [9], [10], [13] (where novel methods for the characterization of the stability domains for saturated feedback systems were employed to provide a systematic design anti-windup design tools), [26] (where the results of [18] were extended to the case of a narrowed sector bound, thus obtaining a locally stabilizing anti-windup compensator), and [12], [11] (which extend the coprime factor based anti-windup solution initially proposed in [21]). For the case of Linear Parameter Varying (LPV) systems (which can be used to embed an uncertain, possibly nonlinear and time varying plant, and then to solve a robust-in-the-large stabilization problem), some solutions have been proposed: see *e.g.* [8], [27] and references therein.

In this paper, we address the anti-windup design for exponentially unstable linear plants using a *nonlinear* anti-windup structure based on the architecture first introduced in [24] and then further developed in [23], [5], [4] for exponentially unstable plants. In particular, we show that the constructive anti-windup solution proposed in [17] can be readily extended in order to give a robust-in-the-large solution to the anti-windup problem for exponentially unstable, uncertain linear plants. The key advantage of the approach in [23] (preserved in [17], and then in the solution proposed in the present paper, even in the presence of uncertainties) is that unlike the previous approaches in [10], [9], [12], [11], [26], the compensation structure is only dependent on the plant dynamics, and then the achievable operating region in the plant state space is only dependent on the structural limitations of the saturated uncertain plant; on the other hand, previously proposed solutions (as the ones cited above) also depend on dynamics of an *a priori* given unconstrained controller (which is part of the anti-windup problem definition), so that, especially when that controller is very aggressive, the corresponding constructions may lead to very small operating regions. As in [17], an important advantage of our technique as compared to the existing ones is that we are able to guarantee bounded responses to references of arbitrarily large size, because the plant state is permanently monitored and kept within the null-controllability region, thus preserving the overall stability property; however, in order to achieve the largest possible operating regions (not achievable by previously proposed LPV approaches), the construction in [17] is modified here by the use of polyhedral Lyapunov functions and related tools, on which we heavily rely [7], [6]. Also, with respect to previous work on anti-windup for LPV systems, no bounds on the rate of variation of parameters are required.

We remark that, although the anti-windup construction in this paper is (up to a few additional technicalities) essentially the one proposed in [17] (and, in particular, the formal proof of our main result is not even reported, being a mere

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repetition of the proof in [17]), the main value of this contribution relies in showing the following facts:

1) the anti-windup solution of [24], [23] (in particular, as implemented in [17]) can be easily extended to LPV (possibly exponentially unstable) systems;
2) the necessary (but not sufficient, in general) condition of unconstrained loop robust stability [25] (and the independent work [15], [14]) is also *sufficient* for LPV anti-windup, so that for this class of systems there is no trade-off between performance and robust-in-the-large stability in the sense of [16];
3) the key role of the knowledge of the time-varying parameters in our anti-windup solution lies *not* in its use for robust stabilization, *but* in the possibility of preserving a so-called "cascade structure" [24] inside the anti-windup closed loop; this structure gives the possibility to reduce the anti-windup problem to a state feedback, constrained stabilization problem for an uncertain LPV system whose dynamics is only dependent on the plant (which can be solved *without* using the knowledge of the time-varying parameters) thus making it possible to achieve the largest possible basin of attraction subject to the intrinsic limitations of the plant.

The paper is structured as follows: after introducing some notation, in Section II the data of the problem are described; in Section III the anti-windup problem of interest is formally defined, explaining how and why the definition in [17] needs to be extended; the anti-windup compensator design is described in Section IV, and finally, we show the effectiveness of the approach on a simulation example in Section V .

Notation Let $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) be the set of positive (non negative) reals. Given $w, v \in \mathbb{R}^p$, the inequality $w > v$ must be understood componentwise, *i.e.* $w > v$ means $w_i > v_i$ for all $i = 1, \dots, p$ (equivalently, $w - v \in \mathbb{R}^p_{\geq 0}$).

The scalar saturation function of unitary level is indicated as

$$\sigma(v) := \begin{cases} \operatorname{sign}(v), & \text{if } |v| > 1; \\ v, & \text{if } |v| \leq 1; \end{cases}$$

where sign$(\cdot)$ is the sign function; the (vector, decentralized) saturation function of unitary level is defined by saying that its i-th component is a scalar saturation function of unitary level.

A signal $q(\cdot)$ is in $\mathcal{L}_p$ if its $\mathcal{L}_p$ norm is bounded, where

$$\|q\|_p := \begin{cases} \left(\lim_{t\to\infty} \int_0^t \|q(\tau)\|\, d\tau\right)^{1/p} & \text{if } p \in [1, \infty), \\ \sup_{\tau\in[0,+\infty)} \|q(\tau)\| & \text{if } p = \infty. \end{cases}$$

The class of piecewise continuous functions of time[1] will be denoted as $\bar{\mathcal{C}}^0$.

A polyhedral subset $\mathcal{P} \subset \mathbb{R}^n$ is a set defined by the relation $\mathcal{P} := \{v \in \mathbb{R}^n : Fv \leq \bar{1}\}$, where $\bar{1} := [1 \cdots 1]'$ and F is a matrix of suitable dimensions. A function $\Psi \in \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a *gauge function* if

- $\forall x \in \mathbb{R}^n$ $\Psi(x) \geq 0$ and $\Psi(0) = 0$,
- $\forall \lambda \in \mathbb{R}_{\geq 0}$, $\forall x \in \mathbb{R}^n$ $\Psi(\lambda x) = \lambda \Psi(x)$,
- $\forall x, y \in \mathbb{R}^n$, $\Psi(x + y) \leq \Psi(x) + \Psi(y)$.

[1]Throughout the paper, piecewise continuity of signals will be assumed for simplicity; this assumption can be relaxed by simply assuming that the relevant signals are sufficiently regular to satisfy the usual sufficient conditions for existence of solutions.

Any compact and convex polyhedral set $\mathcal{P} := \{x \in \mathbb{R}^n : Fx \leq \bar{1}\}$ is associated to the sublevel set $\bar{N}[\Psi, k] = \{x \in \mathbb{R}^n : \Psi(x) \leq k\}$ for $k = 1$ of the gauge function $\Psi(x) = \max_{1\leq i\leq n}(F_i x)$, where F_i denotes the $i-$th row of matrix F; conversely, that any compact and convex set $\mathcal{S}$ induces a gauge function $\Psi(x) = \inf\{\mu \in \mathbb{R}_{\geq 0} : x \in \mu\mathcal{S}\}$.

II. Problem setting

Let $\mathcal{D}_0$ be a convex and compact polyhedron, and $\mathcal{W}_0 := \{w \in \mathbb{R}^\mu_{\geq 0} : \sum_{i=1}^\mu w_i = 1\}$; moreover, define the classes of set bounded, piecewise continuous disturbances $\mathcal{D} = \{d(\cdot) \in \bar{\mathcal{C}}^0 : d(t) \in \mathcal{D}_0\}$ and time-varying parameters $\mathcal{W} = \{w(\cdot) \in \bar{\mathcal{C}}^0 : w(t) \in \mathcal{W}_0\}$.

Consider the linear parameter varying (LPV) plant

$$\dot{x} = A(w)x + B(w)u + Ed, \tag{1a}$$
$$z = C_1(w)x + D_1(w)u + F_1 d, \tag{1b}$$
$$y = C_2(w)x + D_2(w)u + F_2 d, \tag{1c}$$

where $x \in \mathbb{R}^n$ is the plant state, $u \in \mathbb{R}^m$ is the control input, y is the plant output available for measurement and z is the performance output; and each matrix of the form $M(w)$ in (1) is defined in terms of its μ "vertex values" M_i, $i = 1, \dots, \mu$, according to the relation $M(w) = \sum_{i=1}^\mu w_i(t) M_i$. The signal $w(\cdot)$ is assumed to be available (measured).

An *a priori* given "unconstrained controller"

$$\dot{x}_c = A_c(w)x_c + B_c(w)u_c + B_r(w)r, \tag{2a}$$
$$y_c = C_c(w)x_c + D_c(w)u_c + D_r(w)r. \tag{2b}$$

is assumed to be available for plant (1); the name "unconstrained controller" given to (2) is motivated by the assumption that (2) is designed assuming the "unconstrained interconnection"

$$u = y_c, \quad u_c = y, \tag{3}$$

and is such that the unconstrained closed-loop system $\bar{\Sigma}_U$ given by (1), (2), (3) has a desirable response to external signals r, d; this implies that the following (minimal) assumption is satisfied.

Assumption 1: $\bar{\Sigma}_U$ is well-posed and asymptotically stable, $\forall w \in \mathcal{W}$.

Remark 1: According to Assumption 1 (which is a necessary condition for robust anti-windup compensation, as pointed out in [25], [16]), the unconstrained controller (2) guarantees *global robust asymptotic stability* of the unconstrained closed-loop system. However, we remark that matrix $A(w)$ is not assumed to be Hurwitz (even for fixed values of w), and actually the uncertain controlled system (1) is not assumed to be asymptotically stable or time-invariant; this is in contrast with most of the available literature on anti-windup, where the controlled plant is either unstable (but without "large" uncertainties) or affected by large uncertainties (but asymptotically stable). ◦

When saturation is present at the plant input, the unconstrained interconnection (3) is replaced by the following *saturated interconnection*:

$$u_c = y, \quad u = \sigma(y_c). \tag{4}$$

The corresponding *saturated closed-loop system* $\widehat{\Sigma}_S$ given by (1), (2) and (4) typically exhibits undesirable behavior, since the controller (2) is designed without taking into account the saturation constraints; moreover, structural constraints (*i.e.* the fact that for plants with exponentially unstable modes and bounded input the null controllability region is bounded in the exponentially unstable directions; see, *e.g.*, [22]) imply that the global stability properties in Assumption 1 will be lost for $\hat{\bar{\Sigma}}_S$ if (1) is exponentially unstable.

In order to limit the adverse effects of saturation, and to recover the unconstrained responses of $\bar{\Sigma}_U$, an *anti-windup compensator* can be designed; in general, this compensator will be a dynamical system having a state x_{aw} and two output signals v_1 and v_2, used to modify the saturated closed-loop system $\widehat{\Sigma}_S$ according to the relation

$$u_c = y + v_2, \quad u = \sigma(y_c + v_1). \tag{5}$$

The interconnection of (1), (2) and the anti-windup compensator according to (5) will be denoted as the (saturated) anti-windup closed-loop system $\breve{\Sigma}_{SAW}$. As already anticipated, the goal of anti-windup compensation can be qualitatively stated as preserving (*i.e.* not modifying) as much as possible the response of the unconstrained closed-loop $\bar{\Sigma}_U$, meanwhile neutralizing the destabilizing effects of saturation (which, in the case of exponentially unstable plants, include the possibility of leaving the null controllable region); otherwise stated, it is desirable that 1) the response from r, d to z of $\breve{\Sigma}_{SAW}$ must be equal to the corresponding response of $\bar{\Sigma}_U$ (and this happens if $v_1 = 0$ and $v_2 = 0$) as long as no saturation occurs and the state of the plant remains inside (a proper subset of) the null controllable region, and 2) $\breve{\Sigma}_{SAW}$ must be stable.

III. Anti-windup problem definition

In order to formally state the anti-windup problem of interest in this paper, some preliminary definitions are needed.

Let $\mathcal{X}$ and $\mathcal{X}^+$ be two compact and convex subsets of the null controllable region of (1) under the available bounded input, such that $\exists\epsilon > 0$ for which $(1+\varepsilon)\mathcal{X} = \mathcal{X}^+ \subset \mathbb{R}^n$. The proposed design of the anti-windup compensator (6) will guarantee that 1) the state x of (1) never leaves the region $\mathcal{X}^+$, and 2) the response (from r, d to z, for $x_{aw}(0) = 0$) of $\breve{\Sigma}_{SAW}$ will be equal to the corresponding response of $\bar{\Sigma}_U$ as long as no saturation occurs and the state of (1) in $\bar{\Sigma}_U$ remains inside $\mathcal{X}$. Once again, it is remarked that introducing the sets $\mathcal{X}$ and $\mathcal{X}^+$ is necessary in order to achieve the requirements in the next definition (in particular, $\mathcal{X}^+$ for stability and $\mathcal{X}$ for local preservation); the "distance" ϵ between the boundaries of $\mathcal{X}$ and $\mathcal{X}^+$ is needed in order to allow a region where the anti-windup compensator (6) can "brake" in order to avoid the state x to leave $\mathcal{X}^+$. Notice also that, since the boundary of the null controllable region is an invariant set, in order to avoid to loose the ability to quickly steer the state x to points inside $\mathcal{X}$, it is desirable to guarantee some distance between the boundary of $\mathcal{X}^+$ and the boundary of the null controllable region (more on this "stickiness effect" can be found in [3, Remark 5]).

Once the region $\mathcal{X}$ where the state x is supposed to evolve is defined, a set of *(steady state) feasible external signals* (including both references and disturbances) can be defined, containing those pairs of constant references and disturbances leading to equilibria within the set $\mathcal{X}$. This is done in the following definition.

Definition 1: Given a set $\mathcal{X}$ and a signal $w \in \mathcal{W}$, let $\mathcal{RD}(w, \mathcal{X})$ be the *set of feasible external signals for* w *and* $\mathcal{X}$, where the pair $(r_\circ, d_\circ)$ is a feasible external signal for w and $\mathcal{X}$ if the state response of $\bar{\Sigma}_U$ to the external inputs $(r(t), d(t)) = (r_\circ, d_\circ)$, $\forall t \geq 0$, converges to a steady state value (x^*, x_c^*) with $x^* \in \mathcal{X}$. $\circ$

Remark 2: Compared with the corresponding definition in [17], three main differences can be noticed in Definition 1. First, w is not assumed to be constant in Definition 1. Since no parameter variations were allowed in [17] (*i.e.* w was a constant), by Assumption 1 the response of the linear and stable system $\bar{\Sigma}_U$ was always convergent; on the other hand, since in our case w could be any signal in $\mathcal{W}$, there is no *a priori* guarantee that the state response of $\bar{\Sigma}_U$ to constant $(r_\circ, d_\circ)$ will converge, unless w is constant too. However, assuming a constant value of w for this reason is unnecessarily conservative (such condition is only sufficient; it is easy to figure out cases such that w does not even converge to a fixed value, and yet there is an associated constant equilibrium (x^*, x_c^*) of $\bar{\Sigma}_U$), and then only convergence of the state response of $\bar{\Sigma}_U$ is assumed.
Second, the set $\mathcal{RD}(w, \mathcal{X})$ of feasible external signals depends on w (and not only on $\mathcal{X}$, as in [17]). This complication is needed since very little restrictions are imposed on w by the fact the $w \in \mathcal{W}$; hence, considering simply

$$\begin{aligned}\mathcal{RD}(\mathcal{X}) &:= \cap_{w\in\mathcal{W}} \mathcal{RD}(w, \mathcal{X}) \\ &= \{(r_\circ, d_\circ) : (r_\circ, d_\circ) \in \mathcal{RD}(w, \mathcal{X}), \forall w \in \mathcal{W}\} =\end{aligned}$$

(*i.e.* the set of external signals (r, d) for which the state response of $\bar{\Sigma}_U$ converges, for any $w \in \mathcal{W}$) could be overly restrictive, considering only those (r, d) which are admissible for *any* $w \in \mathcal{W}$; on the other hand, the somewhat "implicit" definition given above allows to recover as much as possible any (r, d) that are admissible for the specific $w(\cdot)$ affecting the plant.
Third, both $r_\circ$ and $d_\circ$ are simoultaneously accounted for in the definition of $\mathcal{RD}(w, \mathcal{X})$, so that there is a trade-off between the size of $r_\circ$ and the size of $d_\circ$ in each feasible pair $(r_\circ, d_\circ)$. In [17], $d_\circ = 0$ could always be assumed without any loss of generality due to the particular structure considered for the plant dynamics in [17]; on the other hand, since no similar assumption is made in our paper about the structure of the plant dynamics, the value of $d_\circ$ concurs in determining the steady state value of the state, thus limiting the feasible values of $r_\circ$. $\circ$

For the next definition, given certain selections of $w(\cdot) \in \mathcal{W}$, the external inputs $r(\cdot)$ and $d(\cdot) \in \mathcal{D}$ and initial conditions $x(0), x_c(0)$ for the plant and the unconstrained controller states, we will denote by $\bar{\cdot}$ the responses arising from the unconstrained closed-loop system $\bar{\Sigma}_U$ (e.g., $\bar{x}(\cdot), \bar{u}(\cdot), \bar{z}(\cdot)$ and so on), and by $\breve{\cdot}$ the responses arising from the anti-windup closed-loop system $\breve{\Sigma}_{SAW}$ (e.g., $\breve{x}(\cdot), \breve{u}(\cdot), \breve{z}(\cdot)$ and so on). The following problem will be addressed and solved in this paper.

Definition 2: Given a compact and convex set $\mathcal{X} \subset \mathbb{R}^n$, the *anti-windup problem for* $\mathcal{X}$ is to design an augmentation to the controller (2) such that for any initial condition $x(0), x_c(0)$ satisfying $x(0) \in \mathcal{X}$ and any $w(\cdot) \in \mathcal{W}$, $r(\cdot)$ and $d(\cdot) \in \mathcal{D}$, the corresponding responses of the unconstrained closed-loop system $\bar{\Sigma}_U$ and of the anti-windup closed-loop system $\breve{\Sigma}_{SAW}$ satisfy the following properties:

1) **(local preservation)** if $\sigma(\bar{u}(t)) = \bar{u}(t)$, $\bar{x}(t) \in \mathcal{X}$, $\forall t \geq 0$, then $\breve{z}(t) = \bar{z}(t)$, $\forall t \geq 0$;
2) **($\mathcal{L}_p$ recovery)** $\forall (r_\circ, d_\circ) \in \mathcal{RD}(w, \mathcal{X})$, if $(r(\cdot) - r_\circ, d(\cdot) - d_\circ) \in \mathcal{L}_p$ then $(\breve{z} - \bar{z})(\cdot) \in \mathcal{L}_p$, $\forall p \in [1, \infty]$;
3) **(restricted tracking)** if $\lim_{t\to+\infty} (\bar{x}(t), \bar{x}_c(t)) = (\bar{x}^*, \bar{x}_c^*)$ with $\bar{x}^* \in \mathcal{X}$ then $\lim_{t\to+\infty} (\breve{x}(t), \breve{x}_c(t)) = (\breve{x}^*, \breve{x}_c^*) = (\bar{x}^*, \bar{x}_c^*)$. Moreover, if $\bar{x}^* \notin \mathcal{X}$ but $\lim_{t\to+\infty} d(t) = 0$, then $\lim_{t\to+\infty} (\breve{x}(t), \breve{x}_c(t)) = (\breve{x}^*, \breve{x}_c^*)$ with $\breve{x}^* \in \mathcal{X}$.

$\circ$

The three requirements in Definition 2 are commented upon in the following remark.

Remark 3: The three items in Definition 2, are three desirable properties guaranteed by our construction. Item 1 (local preservation) guarantees that any trajectory generated by the unconstrained closed-loop system that never saturates and never violates the necessary constraint on the operating region $\mathcal{X}$ (therefore being safely reproducible on the saturated plant) will be preserved by the anti-windup compensation scheme. Item 2 ($\mathcal{L}_p$ recovery) guarantees that any unconstrained trajectory generated by a reference-disturbance pair converging (in an $\mathcal{L}_p$ sense) to a feasible reference-disturbance pair will be asymptotically recovered (in an $\mathcal{L}_p$ sense). This property ensures that any unconstrained trajectory which converges to an admissible set point will be recovered by the anti-windup compensation scheme, even if saturation will impose to give up on some transient performance features. Note that this item evaluated for $p = \infty$ imposes that the anti-windup closed-loop is BIBS stable, that is: any (arbitrarily large) selection of the reference-disturbance pair will lead to a bounded response. Finally, item 3 (restricted tracking) guarantees that any converging unconstrained trajectory will correspond to a converging anti-windup trajectory. All trajectories that converge in forbidden regions for the saturated plant will be projected on a restricted set-point such that the unstable part of the plant state remains in $\mathcal{X}$. $\circ$

Remark 4: Note that the main challenge in finding a solution that guarantees the anti-windup property of Definition 2 resides in the fact that the null controllability region of the plant is bounded in the exponentially unstable directions (see, e.g., [22]). Therefore, special care by way of nonlinear functions needs to be taken to keep such "unstable part" of the state (called x_u in [17]) within the null controllable region at all times, otherwise stability couldn't be guaranteed. For this reason in [17] it was assumed that x_u could be exactly measured, and that no disturbance could affect its dynamics, in order to prove general results on the arising closed-loop. We remark that here we allow the disturbance to act on all the state, and assume that the whole state is measured; the reason for this will be clarified in the following Remark 6. However, as in [17], it is worth to remark that even if exact measurement of x are not available, regional result can be obtained, at the price of a reduction of the region $\mathcal{X}$ (in order to introduce an additional "safety boundary"). $\circ$

IV. Anti-windup construction

The key observation in this paper is that if, as usually is the case in gain scheduled LPV control, the plant is described by (1) and the signal $w(\cdot)$ is measured, the plant dynamics can be *exactly* copied; hence, as in [24], if the following anti-windup compensator structure is used:

$$\dot{x}_{aw} = A(w)x_{aw} + B(w)[y_c - \sigma(y_c + v_1)], \tag{6a}$$
$$v_1 = \alpha(x, x + x_{aw}, y_c, w), \tag{6b}$$
$$v_2 = C_2(w)x_{aw} + D_2(w)[y_c - \sigma(y_c + v_1)], \tag{6c}$$

and $\alpha(x, x + x_{aw}, y_c, w)$ is chosen in such a way to make $\mathcal{X}^+$ forward invariant and to stabilize the dynamics (6a), then the overall anti-windup closed loop system satisfies the requirements in Definition 2.

In order to specify how the function $\alpha(x, x + x_{aw}, y_c, w)$ can be designed, recall that the controlled plant is described by (1), where matrices $A(w)$ and $B(w)$ are defined in terms of their μ "vertex values" A_i, B_i, $i = 1, \ldots, \mu$, according to the relation $A(w) = \sum_{i=1}^{\mu} w_i(t)A_i$, $B(w) = \sum_{i=1}^{\mu} w_i(t)B_i$; moreover, the disturbance acting on the plant is such that $d(t) \in \mathcal{D}_0$, $\forall t \geq 0$, (where $\mathcal{D}_0$ is a convex and compact polyhedron), and the input u to the plant is also bounded in the set $\mathcal{U} := \{u \in \mathbb{R}^p : |u_i| \leq 1, i = 1, \ldots, p\}$. The proposed design procedure (based on the results in [7], [6] and corresponding, with some modifications, to the procedure in [17]), is now described.

Procedure 1: anti-windup compensator design.

Step 1. *Compute a polyhedral domain of attraction* $\mathcal{X}^+ := \{x \in \mathbb{R}^n : Fx \leq \bar{1}\}$ *and define the associated gauge function* $\psi(x) := \max_i(F_i x)$ *and the set* $\mathcal{X} := (1 + \varepsilon)^{-1}\mathcal{X}^+$ *for a small* $\varepsilon > 0$.

This step can be easily accomplished by choosing a (sufficiently) small positive constant $\tau > 0$ and parameters $\lambda^* \in (0, 1)$, $\epsilon > 0$ such that $\lambda = \lambda^* + \epsilon < 1$ and using the algorithm in [6] on the system data A_i, B_i, $i = 1, \ldots, \mu$, $\mathcal{D}_0$ and $\mathcal{U}$. Here, λ gives a level of guaranteed convergence speed, and ϵ is a parameter which allows to trade-off accurate approximation of the domain of attraction with the size of the matrix F describing $\mathcal{X}^+$; a smaller ϵ gives a more accurate approximation, and a smaller λ guarantees a faster convergence (and, due to the bounds on the input, a smaller region $\mathcal{X}^+$).

Step 2. *Compute a control law* $\phi(x)$ *making* $\mathcal{X}^+$ *forward invariant*

As shown in [6], this step can be accomplished in two phases: first, the control value on the vertices of $\mathcal{X}^+$ is determined; then, a Lipschitz continuous extension of the control law is designed for the whole $\mathcal{X}^+$ domain.

The first phase is done by finding, for each vertex v of $\mathcal{X}^+$, a control value u_v such that

$$F[(I + \tau A_j)v + \tau B_j u_v] < \lambda\bar{1} - \delta, \quad \forall j \in \{1, \ldots, \mu\},$$

where vector δ has components $\delta_i = \max_d(F_i E d)$.

Hence, the control law at vertex is used to define the continuous control law for all $x \in \mathcal{X}^+$:

- for all state x exists a cone C_h such that $x \in C_h$, where the vertices of C_h are $X^{(h)} = \begin{bmatrix} x_1^{(h)} & \ldots & x_n^{(h)} \end{bmatrix}$ and the control values associated are $U^{(h)} = \begin{bmatrix} u_{x_1^{(h)}} & \ldots & u_{x_n^{(h)}} \end{bmatrix}$
- the control can be defined as a combination of the values at vertex, then $\phi(x) = U^{(h)}[X^{(h)}]^{-1}x$.

Step 3. *Define the pseudo-tracking control law* $\Phi(x, \bar{x}, \bar{u})$

This step can be accomplished by defining

$$\Psi(x, \bar{x}) := \max_i \frac{F_i(x - \bar{x})}{1 - F_i\bar{x}}, \qquad \tilde{x}(x, \bar{x}) := \bar{x} + \frac{(x - \bar{x})}{\Psi(x, \bar{x})},$$

and the pseudo-tracking control law as

$$\Phi(x,\bar{x},\bar{u}) := \phi(\tilde{x}(x,\bar{x}))\Psi(x,\bar{x}) + (1-\Psi(x,\bar{x}))\bar{u}.$$

As shown in [7], when $A(w) = A$ and $B(w) = B$ are constant matrices and the pair $(\bar{x},\bar{u})$ corresponds to an equilibrium (namely, $A\bar{x} + B\bar{u} = 0$), the control law $\Phi(x,\bar{x},\bar{u})$ ensures asymptotic stability of $\bar{x}$ with domain of attraction $\mathcal{X}$; here, $(\bar{x},\bar{u})$ do not, in general, satisfy such an assumption, hence the name of pseudo-tracking control law.

Step 4. *Define the anti-windup control law* $\alpha(x, x_M, y_c, w)$ This step can be accomplished by defining

$$\alpha(x, x_M, y_c, w) := -y_c + \Phi(x, \pi(x_M), \pi_u(y_c, w))$$

where

$$\pi(x_M) := \begin{cases} x_M & \text{if } x_M \in \mathcal{X} \\ \dfrac{x_M}{\psi(x_M)(1+\epsilon)} & \text{if } x_M \notin \mathcal{X} \end{cases}$$

$$\pi_u(y_c, w) := \begin{cases} y_c & \text{if } y_c \in \mathcal{U} \text{ and } x_M \in \mathcal{X} \\ -B(w)^\sharp A(w) p(x_M) & \text{otherwise} \end{cases}$$

and $B(w)^\sharp := (B(w)'B(w))^{-1}B(w)'$.

The effectiveness of the above procedure is stated in the following theorem, whose proof is omitted since it coincides with the proof in [17] (up to some minor technical modifications due to the use of polyhedral Lyapunov functions, and a slightly simpler definition of the functions appearing in Step 4).

Theorem 1: Under Assumption 1, the anti-windup compensator designed according to Procedure 1 solves the problem in Definition 2.

Remark 5: The LPV system (1) can be used to "hide" a nonlinear, uncertain and time varying system under a "linear" structure, provided that the linearizations of the considered system in the operating region of interest are all contained in the convex hull of the vertex values A_i, B_i, ... defining the matrices $A(w)$, $B(w)$, ... (possible drift terms due to the fact that not all the states inside the considered operating region are equilibria can be modeled by absorbing them in the "disturbance" d).
In this regard, it is relevant to remark that such procedure is obviously conservative (since, for example, in the original nonlinear system some "parameter variations" of the LPV system will not be possible; a similar remark applies with respect to the fact that no bound on the rate of variation of $w(\cdot)$ is imposed in our analysis); however, it is important to understand that *the proposed result is tight for the LPV system* (1), since (up to an arbitrarily small constant ε) the largest possible domain of attraction under the given bounds on the control input, the disturbances and the uncertainties is obtained by the construction in Procedure 1 by virtue of the algorithms in [6]. ◦

Remark 6: In [17] (following [23]) a partition of the state of the plant as $x := [x_s' \ x_u']'$ was considered, and the plant dynamics was consistently partitioned as

$$\begin{bmatrix} \dot{x}_s \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} u + \begin{bmatrix} B_{ds} \\ 0 \end{bmatrix} d, \quad (7)$$

where A_s is Hurwitz. For the complete motivation of such a choice, the reader is referred to [23]; what is important to notice here is that an useful consequence of such an assumption (exploited in [23], [3], [17]) is that since A_s is Hurwitz in (7), the null controllable region of the plant is unrestricted in the x_s part of the state, so that only measurements of x_u are needed by the anti-windup controller in order to preserve boundedness of trajectories. However, when time-varying parameters are present, and unless specific structural (*i.e.* non generic) assumptions hold about the dependence of the system's state space description on the unknown parameters, the null controllable region "moves" as the parameters vary with time, and then it is not possible to measure only the "unstable part" x_u of the state, and then the whole state must be measured; this was shown by an example in [2], where the same aircraft model used in [3] was studied, in the presence of parameter variations. For this reason, the whole state x is assumed to be measurable in this paper. ◦

V. Simulation examples

In this section, the example in [17] Consider the plant described by matrices

$$A(\omega_1) = \omega_1 \begin{bmatrix} 1.8 & -1 \\ -0.2 & 0.8 \end{bmatrix} + (1-\omega_1)\begin{bmatrix} 2.2 & -1 \\ 0.2 & 1.2 \end{bmatrix}$$

$$B(\omega_2) = \omega_2 \begin{bmatrix} 9.8 \\ -6.8 \end{bmatrix} + (1-\omega_2)\begin{bmatrix} 10.2 \\ -7.2 \end{bmatrix}$$

with $C_1 = [\ 1 \quad 1\ ]$, $C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E = D_1 = D_2 = 0$.

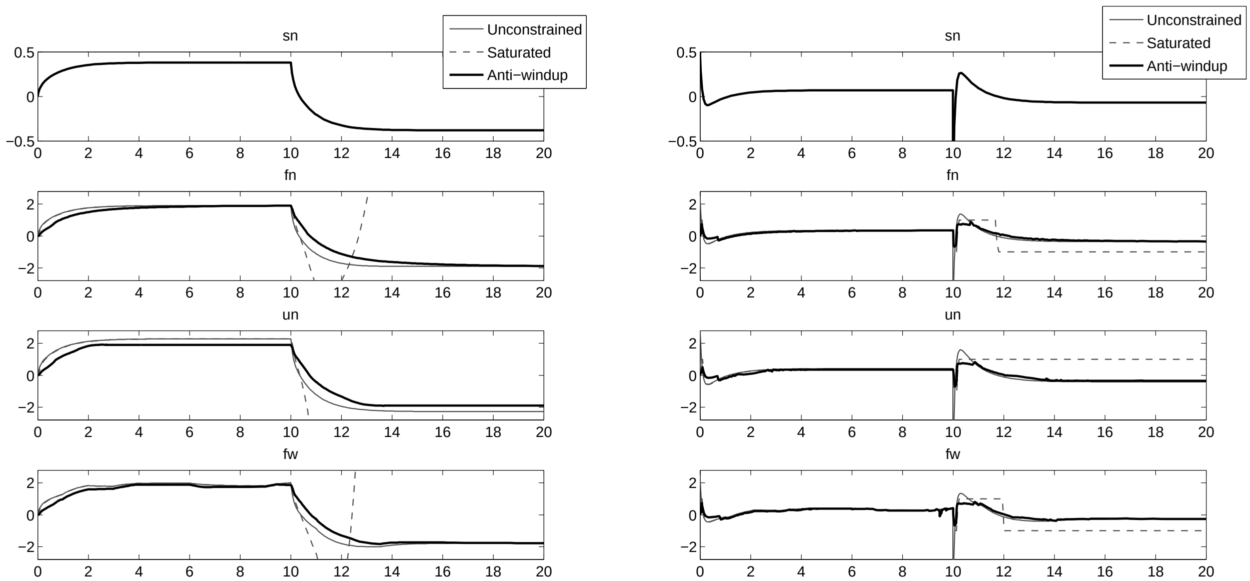
The proposed design procedure has been applied with $\lambda^* = 0.982$, $\epsilon = 0.012$, and A_0, B_0 corresponding to the parameter values $(\omega_1, \omega_2) = (0.5, 0.5)$.

The unconstrained controller, ensuring robust asymptotic stability in the absence of saturation, has been obtained by using the LQR technique with $R = 1$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The reported figures show the performance output and control input for the unconstrained closed loop system ($\bar{\Sigma}_U$), for the saturated closed loop system ($\hat{\Sigma}_S$), and for the saturated anti-windup system ($\tilde{\Sigma}_{SAW}$). In the three upper subplots of each figure, the nominal parameter values ($\omega_1, \omega_2 = 0.5, 0.5$) are considered, and the letters *sn* denote the response to a small reference, *fn* denote the response to a feasible reference close to the largest feasible reference r_{MAX} and *un* denote the response to an unfeasible reference. In the last subplot (identified by the letters *fw*), a time-varying parameter signal $w(\cdot)$ is considered, coupled with a large feasible reference close to r_{MAX}.

In each case, it can be notice that the state of the saturated closed loop system leaves the null controllable region and then its output diverges, whereas the forward invariance of the set $\mathcal{X}^+$ guaranteed by our anti-windup compensator preserves the anti-windup closed loop from diverging. Moreover, whenever the reference is feasible (cases *sn* and *fn*), the output of the anti-windup closed loop converges to the output of the unconstrained closed loop; when the reference is not feasible (case *un*), the output of the anti-windup closed loop still converges, and reaches a value close to the output of the unconstrained closed loop (but such that the state x remains inside the set $\mathcal{X}$).

Finally, notice also from case *fw* that, as specified in the anti-windup problem definition, that the output of the anti-windup closed loop is close to tracking the output of the unconstrained closed loop system during the first seconds (*i.e.* while the parameter signal $w(\cdot)$ keeps varying), and then



(a) Performance output signal. Comparison among the closed loop response for small reference, large feasible reference, unfeasible reference and large feasible reference with perturbations

(b) Control signal. Comparison among the closed loop response for small reference, large feasible reference, unfeasible reference and large feasible reference with perturbations

(when $w(\cdot)$ stops varying), the output of the unconstrained closed loop system eventually converges to a constant value and is reached by the output of the anti-windup closed loop.

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