A family of global stabilizers for quasi-optimal control of planar linear saturated systems

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***Abstract*—We propose a family of nonlinear state feedback global stabilizers for all planar linear systems which are globally stabilizable by bounded inputs (namely, all non exponentially unstable linear systems). This family is parametrized by a nonlinear function whose selection can yield quasi time-optimal responses, where the "quasi" is required to achieve local exponential stability of the closed-loop. The arising trajectories are quasi time-optimal for arbitrarily large initial conditions; so, we expect the very simple proposed nonlinear control law to be very useful for embedded control applications with strong computational constraints.**

***Index Terms*—Saturation, time-optimal control, fuel-optimal control, global stabilization, Lyapunov functions.**

I. Introduction

Planar systems with input saturation have been long studied in the control literature. For experimental purposes, planar models are often already sufficient to characterize the main dynamic behavior of a wide family of plants, so that high performance control laws arising from studies on planar systems might become very effective in several applications (see, e.g., the case studies mentioned in [20] or the application in [13]). In light of saturation, when designing controllers for these systems quite often one seeks for solutions of the time-optimal (or bang-bang) type, so that the control input authority is fully exploited most of the time.

While there are several valuable studies on time-optimal control of nonlinear planar systems (see e.g. [25], [3], [4] and references therein), we focus here on linear saturated systems. For this class of systems global asymptotic stabilization can only be achieved if the plant is ANCBI (asymptotically null-controllable with bounded inputs), i.e. if its poles are in the closed left half plane [24], [21]. Linear ANCBI systems with at least one imaginary pole can only be globally *asymptotically* (not exponentially) stabilized, and then performance is a key issue when designing a global stabilizer, since one cannot achieve a global exponential convergence rate. While a linear saturated feedback cannot globally stabilize already a triple integrator [11], [26], it can always stabilize a planar linear system [9]. For additional recent results about control of saturated planar systems see e.g. [19], [6], [12], [15], [20].

In this paper we propose a family of static nonlinear controllers for ANCBI planar linear systems which are Lipschitz and extremely close to being time-optimal (or fuel-optimal), thus yielding quasi optimal responses for all signal ranges (which is typically hard to obtain with linear solutions) while preserving the robustness properties of a Lipschitz state-feedback. Though the idea of get a Lipschitz feedback by locally modifying a discontinuous law is well known in the variable structure control literature (see e.g. [23]), the approach in this paper is different (and complementary) in many respects. Furthermore, since the arising control laws are constant in a very large portion of the state space, they seem especially favourable from the point of view of requiring low control attention (in the sense of [5]).

The paper is structured as follows: in Section II we introduce a family of state feedback stabilizers parametrized by a nonlinear function which needs to satisfy a simple gradient condition. In Section III we discuss useful selections of this parameter leading to quasi-optimal responses for certain classes of systems. In Section V we provide several examples illustrating the advantages of the proposed stabilizers. Finally, Section VI contains the proof of the main theorem.

Notation. $\partial_{x_i} f$ denotes the partial derivative of $f(\cdot)$ with respect to x_i. A function $\eta(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is in class $\mathcal{K}$ if $\eta(0) = 0$ and it is strictly increasing; $\eta(\cdot)$ is in class $\mathcal{K}_\infty$ if $\eta(\cdot) \in \mathcal{K}$ and $\lim_{s\to+\infty} \eta(s) = +\infty$.

II. A family of state feedback stabilizers

Consider the following linear planar saturated system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a_1 x_1 - a_2 x_2 + \mathrm{sat}_M(u), \tag{1}$$

where $\mathrm{sat}_M(\cdot)$ is the symmetric scalar saturation function with saturation limits $\pm M$ and u is the control input.[1] We will make the following assumption on (1) throughout this paper.

Assumption 1: The linear plant (1) is globally stabilizable from u, namely (see, e.g., [24], [21]) $a_1,\, a_2 \geq 0$.

We will study in this paper the design of a (nonlinear in general) static state feedback stabilizer for (1) in the form:

$$u = -k\beta(x), \tag{2}$$

where k is a positive constant and $\beta(\cdot)$ is a suitable nonlinear function. This specific choice of feedback with k factored out is motivated by the special form of feedbacks considered in this paper (see also the statement in the following Theorem 1).

For the static controller (2), we will give several recipes in this paper, geared toward the achievement of almost time-optimal and (possibly) fuel-optimal responses. Moreover, we will allow in several cases to enforce an arbitrary local linear behavior on the tail of the closed-loop responses (namely in a suitable neighborhood of the origin). To this aim, it is useful to formally define here the set of functions $\beta(\cdot)$ which are guaranteed to induce desirable stability and convergence properties on the closed-loop, as formally stated in the following assumption and theorem.

Assumption 2: The function $\beta(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ in (2) is a locally Lipschitz function satisfying $\beta(0) = 0$ and

1) there exists $\eta(\cdot) \in \mathcal{K}$ such that
   - if $x_1 \geq 0$ and $x_2 \geq 0$, then $\beta(x) \geq \eta(|x|)$;
   - if $x_1 \leq 0$ and $x_2 \leq 0$, then $\beta(x) \leq -\eta(|x|)$;
2) $\partial_{x_2}\beta(x) \geq 0$ a.e in $\mathbb{R}^2$ and there exists an open set $\mathcal{A}$ such that $0 \in \overline{\mathcal{A}}$ and such that $\partial_{x_2}\beta(x) > 0$ a.e. in $\mathcal{A}$.

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[1] To keep the discussion simple, we will assume symmetric saturations in this paper, however it is possible to extend the results here presented to the case with non-symmetric saturations.

Remark 1: (Interpretations of Assumption 2) The intuitive meaning of Assumption 2 is that the state feedback (2) should preserve the equilibrium at the origin (namely $\beta(0) = 0$) and that β is strictly positive on the first (respectively, strictly negative on the third) closed quadrant take away the origin (item 1). Finally, the constraint on the derivative of β with respect to x_2 at item 2 provides a sufficient condition to guarantee that the β does not induce new equilibria nor limit cycles. The peculiar requirement on the set $\mathcal{A}$ (namely that it is open and its closure contains the origin) is motivated by the fact that for any (arbitrarily small) neighborhood of the origin, we need the strict inequality to hold in a set of positive measure, contained in that neighborhood (see the proof of Theorem 1 for details). ◦

Theorem 1: Given the plant (1), if the function $\beta(\cdot)$ in the control law (2) satisfies Assumption 2 and k is such that

$$k > \inf_{s>0} \frac{M}{\eta(s)} = \lim_{s\to+\infty} \frac{M}{\eta(s)}, \tag{3}$$

for the function $\eta(\cdot)$ in the same Assumption 2, then:

1) all trajectories of (1), (2) converge to the origin;
2) moreover, if $\beta(\cdot)$ is differentiable at the origin and

$$\partial_{x_1}\beta(0) > -a_1, \ \partial_{x_2}\beta(0) > -a_2,$$

then the origin is a locally exponentially stable and globally asymptotically stable equilibrium point.

Proof: See Section VI ■

Remark 2: (Considerations on Theorem 1) The following considerations can be made about Theorem 1:

• *Lower bound on the gain k.* Whenever $\eta(\cdot) \in \mathcal{K}_\infty$, the lower bound on k enforced by (3) is zero. Namely, in this case u can be scaled down to become arbitrarily small in any compact neighborhood of the origin.

• *Global exponential stability.* If the plant (1) is exponentially stable (i.e., both a_1 and a_2 are strictly positive), then under the conditions at item 2 of Theorem 1 (which in this case reduce to $\partial_{x_1}\beta(0) > -a_1$ because of item 2 of Assumption 2), global exponential stability of the closed-loop can be proven by relying on the two global and local exponential bounds.

• *Linear saturated feedback.* Theorem 1 yields global asymptotic (and local exponential) stability of the origin under linear saturated state feedback of the form $\beta(x) = \alpha_1 x_1 + \alpha_2 x_2$, with $\alpha_1 > 0$ and $\alpha_2 > 0$ (which satisfies Assumption 2). This type of feedback has been studied in many papers (e.g., [10] and references therein, [27], [8], [9], [20], [22], [16]).

• *Extension to the nonlinear case.* An extension of Theorem 1 holds under very mild assumptions on $f(x_1, x_2)$ for the nonlinear system with bounded input

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(x_1, x_2) + \mathrm{sat}_M(u). \tag{4}$$

As pointed out in Remark 7, such extension can be used to design quasi time-optimal control laws for (4). ◦

Remark 3: (Robustness properties from Lipschitz continuity) Note that ensuring that the proposed controller is locally Lipschitz guarantees useful robustness properties on the nonlinear closed-loop. As a matter of fact, if the conditions of item 2 of Theorem 1 are satisfied, so that GAS holds, then the results in [28, Theorem 2] guarantee that 1) there exists a smooth converse Lyapunov function for the closed-loop system and, as a consequence, 2) the global asymptotic stability property is robust in the sense that the system can tolerate a suitable perturbation of the dynamics via inner (namely, measurement errors) and outer (namely, actuation errors) inflations (see, [28, Definition 8] for details). ◦

III. Design for quasi time-optimal responses

Some well known facts (cfr [1, Ch. 6-7]) about time-optimal control of planar ANCBI linear systems are now recalled.

A. Time-optimal feedback laws

Time-optimal inputs (ensuring convergence of the state to zero in minimum time) for linear plants with bounded inputs are bang-bang (i.e. can only assume the maximum and minimum allowed values) and can be expressed as a state feedback $u(x)$ defined in terms of a suitable switching surface, described either as $\{x : x_1 + \alpha(x_2) = 0\}$, in which case

$$u = -M\mathrm{sgn}\left(x_1 + \alpha(x_2)\right), \tag{5}$$

or as $\{x : x_2 + \bar{\alpha}(x_1) = 0\}$, in which case

$$u = -M\mathrm{sgn}\left(x_2 + \bar{\alpha}(x_1)\right). \tag{6}$$

For example, for a double integrator with control input bounded between $\pm M$ the optimal feedback is given by

$$u = -M\mathrm{sgn}\left(x_1 + \tfrac{1}{2M} x_2 |x_2|\right), \tag{7}$$

and in this case the function $\alpha(\cdot)$ in (5) is locally Lipschitz, strictly increasing and such that $s\alpha(s) > 0$, $\forall s \neq 0$, i.e. $\alpha(\cdot)$ lies strictly in the first and third quadrant (note that this implies $\alpha(0) = 0$ so that the equilibrium at the origin is preserved). For a harmonic oscillator ($a_1 = \omega^2$, $a_2 = 0$) with control input bounded between $\pm M$ the optimal feedback is given by

$$u = -M\mathrm{sgn}\left(x_2 + \bar{\alpha}(x_1)\right), \tag{8a}$$

$$\bar{\alpha}(x_1) = \tfrac{M}{\omega}\mathrm{sgn}\left(x_1\right)\sqrt{1 - \left(\tfrac{\omega^2}{M}x_1 - 2\left\lfloor \tfrac{\omega^2}{2M}x_1 \right\rfloor - 1\right)^2}, \tag{8b}$$

where $\lfloor s \rfloor$ denotes the integer part of s (i.e. the integer h which is closest to s and such that $|h| \leq |s|$), and in this case the function $\bar{\alpha}(\cdot)$ in (6) is such that $s\bar{\alpha}(s) \geq 0$, $\forall s$, i.e. $\bar{\alpha}(\cdot)$ lies in the closed first and third quadrant (see Fig. 1); notice however that in this case the function $\bar{\alpha}(\cdot)$ is neither monotonically increasing nor locally Lipschitz (the Lipschitz property does not hold at $x_2 = 0$, $x_1 = \frac{2M}{\omega^2}h$, $h \in \mathbb{Z}$). Also for general ANCBI planar system the switching curve lies in the closed first and third quadrant, and then the proposed approach (based on Theorem 1) can be applied.

B. Quasi time-optimal, locally linear Lipschitz feedback

As pointed out before, the advantage of having a locally Lipschitz (instead of a discontinuous, bang-bang) feedback consists in better robustness to noise and disturbances (see Remark 3); moreover, in a neighborhood of the origin it is desirable to have a linear control law in order to have at least local exponential stability.

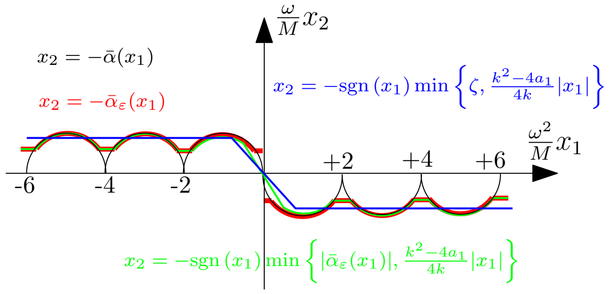

Fig. 1. Normalized time-optimal switching curve for the harmonic oscillator and its approximations: $x_2 = -\bar{\alpha}(x_1)$ (black), $x_2 = -\bar{\alpha}_\varepsilon(x_1)$ (red).

If the optimal feedback law has the form (5), and $\alpha(\cdot)$ is a strictly increasing Lipschitz function, then the function $\beta(x_1, x_2) = x_1 + \alpha(x_2)$ satisfies Assumption 2 and then using this $\beta(x_1, x_2)$ in (2) yields a Lipschitz controller ensuring global convergence according to Theorem 1 (this is the case for the time-optimal control of the double integrator, see (7) and the subsequent comments). However, in a neighborhood of the origin, the above nonlinear selection of $\beta(x_1, x_2)$ can induce a highly oscillatory behavior; hence, it is advisable to introduce a local linear feedback inducing a critically damped local response. The Lipschitz nonlinear control law and the local linear one can be blended by choosing[2] $\beta(x_1, x_2) = x_1 + \mathrm{sgn}\,(x_2)\max\left\{|\alpha(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}$, so that (2) becomes:

$$u = -k\left(x_1 + \mathrm{sgn}\,(x_2)\max\left\{|\alpha(x_2)|, \frac{2}{\sqrt{k}}|x_2|\right\}\right). \qquad (9)$$

It is easy to see that $\mathrm{sat}_M(u)$ with u given by (9) and the time-optimal feedback (5) coincide everywhere except for a stripe around the curve $x_1 + \alpha(x_2) = 0$ having width $\frac{2}{k}$ in the x_1 direction; hence, by increasing k, the above feedback can be made arbitrarily close to the optimal one (at the price of monotonically increasing the locally Lipschitz constants around the switching region – indeed increasing k one becomes closer and closer to the discontinuous law). In particular, for the case of the double integrator (9) becomes

$$u = -k\left(x_1 + x_2\max\left\{\frac{|x_2|}{2k}, \frac{2}{\sqrt{k}}\right\}\right). \qquad (10)$$

Wholly similar comments hold if the optimal feedback law has the form (6), and the selection $\beta(x_1, x_2) = x_2 + \bar{\alpha}(x_1)$ is made, provided that $\alpha(\cdot)$ is a Lipschitz function contained in the first and third quadrant and such that $\liminf_{|s|\to+\infty}|\bar{\alpha}(s)| > 0$ (this condition is weaker than requiring $\bar{\alpha}(\cdot)$ to be strictly increasing). Now, in order to highlight an additional possible obstruction and its solution, consider again the optimal feedback (8) for the harmonic oscillator. Although the just stated condition on $\bar{\alpha}(\cdot)$ is weaker than the one required before, it is clear that the function $\bar{\alpha}(x_1)$ considered in the time-optimal feedback law (8) for the harmonic oscillator does not respect this condition: in fact, $\bar{\alpha}(x_1)$ is zero for $x_1 = \frac{2M}{\omega^2}h$, $h \in \mathbb{Z}$ (hence $\liminf_{|x_1|\to+\infty}|\bar{\alpha}(x_1)| = 0$) and is not Lipschitz at the same points (see Fig. 1). However, both problems can be overcome by a blending which is slightly more general than the one in (9), where in addition to introduce a linear behavior in a neighborhood of the origin, the general time optimal switching curve $x_2 + \bar{\alpha}(x_1) = 0$ is modified as $x_2 + \bar{\alpha}_\varepsilon(x_1) = 0$, where $\bar{\alpha}_\varepsilon(x_1) = \mathrm{sgn}\,(x_1)\max\{|\bar{\alpha}(x_1)|, \varepsilon\}$, in order to exclude a neighborhood of each point where the Lipschitz property is violated (see Fig. 1). The overall arising formula for the harmonic oscillator is

$$u = -k\left(x_2 + \mathrm{sgn}\,(x_1)\min\left\{|\bar{\alpha}_\varepsilon(x_1)|, \frac{k^2-4a_1}{4k}|x_1|\right\}\right), \quad (11)$$

where ε is a sufficiently small positive constant (in particular, $\varepsilon \in \left(0, \frac{M}{\omega}\right)$) and $\bar{\alpha}(x_1)$ is given by (8b). It is easy to see that, as was the case with (9), also (11) is quasi-optimal, in the sense that $\mathrm{sat}_M(u)$ with u given by (11) and the time-optimal feedback (8) coincide except on a stripe around the curve $x_2 + \bar{\alpha}(x_1) = 0$ having width proportional to k^{-1} in the x_2 direction; hence, by increasing k, the above feedback can be made arbitrarily close to the optimal one (at the price of monotonically increasing the associated local Lipschitz constants, as also commented above for the double integrator case). Comparing the feedback law (11) for the harmonic oscillator and the law (10) for the double integrator, it is evident that the implementation of (11) is more complex than the implementation of (10); however, replacing (11) by

$$u = -k\left(x_2 + \mathrm{sgn}\,(x_1)\min\left\{\zeta, \frac{k^2-4a_1}{4k}|x_1|\right\}\right), \qquad (12)$$

(where for the harmonic oscillator a typical choice of the parameters would be $k > 2\omega$ and $\zeta = M\left(\frac{1}{\omega} - \frac{1}{k}\right)$) leads to a much simpler law which also guarantees global attractiveness of the origin (due to Theorem 1), and better global performance than any linear stabilizing law (this is easily seen by comparing the regions where the two kinds of feedbacks differ from the time-optimal feedback law).

Remark 4: (Quasi-optimality, and performance-simplicity trade-off) The above discussion has highlighted that the proposed approach leads to *quasi-optimal control laws*, since it allows to recover the optimal control feedback on all $\mathbb{R}^2$ apart from a small stripe around the curve $\beta(x) = 0$ whose width is a decreasing function of k, converging to a set of measure zero when k goes to infinity.

Another useful feature of the approach is that it allows for a *trade-off between optimality and simple implementation*. In fact (again, cfr classic books as [1]), the exact switching surfaces can be rather complex, and for ease of implementation it can be desirable to choose a simpler curve as the set where $\beta(x) = 0$; as long as the corresponding $\beta(x)$ satisfies Assumption 2, the above approach yields global asymptotic and local exponential stability of the origin. ◦

Remark 5: (Robust convergence and nominal performance) It is perhaps useful to stress that the proposed approach leads to *robust global asymptotic stabilization*, in the sense that, as far as the considered system is in the form (1) (namely, it has relative degree 2 from u to x_1, or, stated otherwise, the first equation preserves the kinematic nature $\dot{x}_1 = x_2$, so that x_1 can be interpreted as position and x_2 as velocity), the proposed control law will still guarantee global asymptotic stabilization, even if the values of $a_1 \geq 0$, $a_2 \geq 0$ are not the nominal values considered during the design stage; moreover,

[2]For simplicity, we omit the dependence on k of $\beta(x_1, x_2)$; the role of such dependence is clear from (9).

quasi-optimality will be achieved if those parameters have (or if they are very close to) their nominal values, and the function $\beta(x_1, x_2)$ and the parameter k have been chosen in order to recover the optimal feedback law. $\circ$

Remark 6: (Step reference tracking and regulation) When at least one of the two eigenvalues of the plant (1) is zero (i.e. whenever $a_1 = 0$), the optimal feedback law for regulating the state to zero also provides the optimal feedback law for tracking the step reference $r(t) = \bar{r}$, $\forall t \geq 0$, provided that $\beta(x_1, x_2)$ is replaced by $\beta(x_1 - \bar{r}, x_2)$. In fact, via the change of variables $\bar{x}_1 = x_1 - \bar{r}$, $\bar{x}_2 = x_2$, the above tracking problem is easily seen to be equivalent to the problem of regulating to zero the new state $\bar{x}_1$, $\bar{x}_2$. $\circ$

Remark 7: (Extension to nonlinear systems) As pointed out in [2], the time-optimal feedback laws for the nonlinear system (4) are topologically equivalent to the time-optimal feedback laws for the linear system (1), provided that $f(x_1, x_2) \in \mathcal{C}^3$ and that $f(x_1, 0) = \pm 1$ implies $\partial_{x_1} f(x_1, 0) \neq 0$. Using the nonlinear extension of Theorem 1 (cfr Remark 2), the approach described in this section can be used to design quasi time-optimal control laws for the nonlinear system (4) too. $\circ$

IV. Design for quasi fuel-optimal responses

Some well known facts about fuel-optimal control of planar ANCBI linear systems are now recalled (cfr [1, Ch. 6 and 8]).

A. Fuel-optimal feedback laws

When the objective is to minimize fuel consumption, optimal solutions may not exist in relevant cases (there exist open regions in the state space such that given an initial state in such a region and any control input ensuring convergence to the origin from that state, it is possible to find a different control input achieving convergence with less fuel, though in a longer time). In order to avoid such situations, it is necessary to bound the maximum allowed transfer time: so, if $T_m(x_0)$ is the transfer time for x_0 under time-optimal control,

• *fixed response time* fuel-optimal inputs guarantee that, for a given $\bar{T} > 0$, the transfer from x_0 is achieved with minimum fuel expenditure in at most $\bar{T}$ time units if $T_m(x_0) \leq \bar{T}$ (i.e. if x_0 is "close" to the origin) or in $T_m(x_0)$ time units otherwise;

• *bounded response time* fuel-optimal inputs guarantee that, for a given $\gamma > 1$, the transfer from x_0 is achieved with minimum fuel expenditure in at most $\gamma T_m(x_0)$ time units.

Similarly to time-optimal inputs, (fixed or bounded response time) fuel-optimal inputs for linear time invariant systems are bang-off-bang (i.e. can only assume the maximum and minimum allowed values, plus the zero value), and can be expressed as a state feedback defined in terms of a suitable switching surface; for example, for a double integrator with control input bounded between $\pm M$ the switching surfaces in the bounded response time case are given by $\left\{x : x_1 + \frac{1}{2M} x_2 |x_2| = 0\right\}$ and $\left\{x : x_1 + \frac{m_\gamma}{M} x_2 |x_2| = 0\right\}$, where $m_\gamma = \frac{\gamma}{2\gamma - 2\sqrt{\gamma(\gamma-1)} - 1} - \frac{1}{2}$ [1, Sec. 8.7, eq. (8-213)] and the optimal feedback is given by

$$u = \begin{cases} -\mathrm{sgn}\left(x_1 + \frac{m_\gamma}{M} x_2 |x_2|\right), & \text{if } x_1(x_1 + \frac{m_\gamma}{M} x_2 |x_2|) \geq 0, \\ -\mathrm{sgn}\left(x_1 + \frac{1}{2M} x_2 |x_2|\right), & \text{if } x_1(x_1 + \frac{1}{2M} x_2 |x_2|) \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

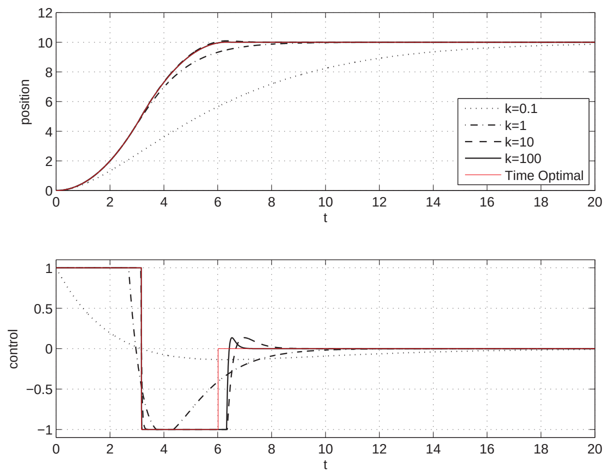

Fig. 2. $M = 1$, quasi time-optimal strategy.

B. Quasi fuel-optimal, locally linear Lipschitz feedback

The key ideas are similar to those expressed in Section III-B; for brevity, the discussion will be limited to the case when the switching surfaces of interest can be expressed as $\{x : x_1 + \alpha(x_2) = 0\}$, $\{x : x_1 + \tilde{\alpha}(x_2) = 0\}$ with $\alpha(s)$ and $\tilde{\alpha}(s)$ two strictly increasing Lipschitz functions lying in the first and third quadrant and such that

$$s\tilde{\alpha}(s) \geq s\alpha(s) > 0, \quad \forall s \neq 0;$$

this is the case for the fuel optimal (with either bounded or fixed response time) solution for the double integrator or plants having one null eigenvalue and one negative eigenvalue.

Defining the following functions:

$$\hat{\alpha}(s) := \max\{\tilde{\alpha}(s), \alpha(s)\}, \quad \check{\alpha}(s) := \min\{\tilde{\alpha}(s), \alpha(s)\},$$
$$\xi(\gamma, s) := \mathrm{sgn}\,(s) \max\left\{|\gamma(s)|, \tfrac{2}{\sqrt{k}}|s|\right\},$$

and using the same blending approach already used in the first part of Section III-B, the proposed control law is

$$u = \begin{cases} -k\left(x_1 + \xi(\check{\alpha}, x_2)\right), & \text{if } x_1 + \xi(\check{\alpha}, x_2) \geq 0, \\ -k\left(x_1 + \xi(\hat{\alpha}, x_2)\right), & \text{if } x_1 + \xi(\hat{\alpha}, x_2) \leq 0, \\ 0, & \text{otherwise,} \end{cases} \tag{13}$$

that is $u = -k \min\{x_1 + \xi(\check{\alpha}, x_2), \max\{x_1 + \xi(\hat{\alpha}, x_2), 0\}\}$.

V. Examples

Double integrator. Constant reference tracking for a double integrator plant with saturated input ($M = 1$) is solved by the quasi-optimal control proposed is (10), blending the bang-bang feedback with a stabilizing linear feedback (thus recovering the time-optimal behavior for large error signals and avoiding problems near the equilibrium due to noise and disturbances). Simulations results are shown in fig. 2 for different values of k. Note that increasing k implies a better recovery of the optimal response and the use of a more aggressive, critically damped (due to the constant $\frac{2}{\sqrt{k}}$ inside the max) linear control law. Theorem 1 guarantees global asymptotic stability for any parameter variation that preserves the kinematic relation $\dot{x}_1 = x_2$ (see Remark 5).

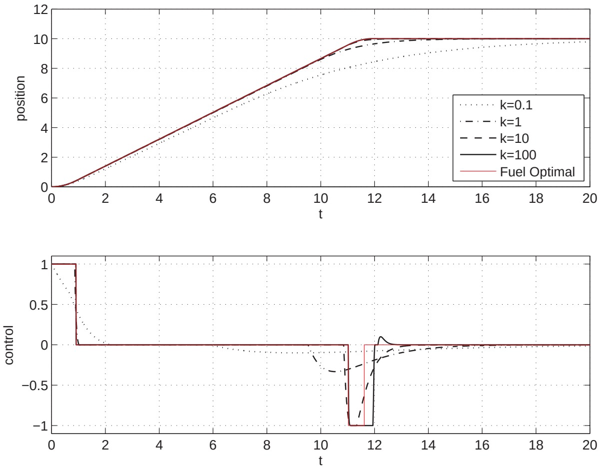

Fig. 3. $M = 1$, $\gamma = 2$, quasi fuel-optimal strategy.

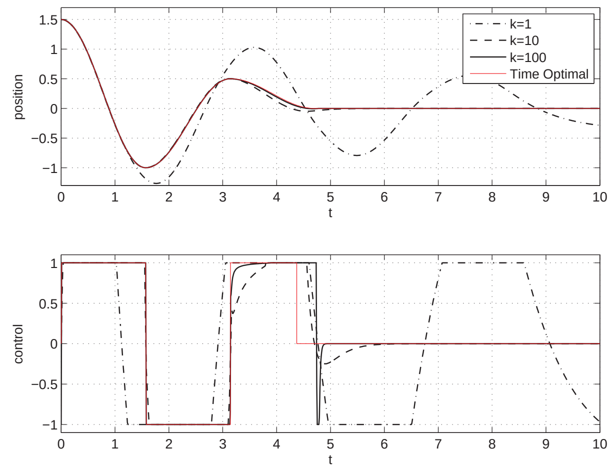

Fig. 4. $M = 1$, $\omega = 2$, quasi time-optimal strategy.

Similarly, fig. 3 shows the responses for different values of k in the quasi fuel-optimal control law given by (13). With $m_\gamma \approx 11.66$ (i.e. $\gamma = 2$) the fuel-optimal response converges in a time no longer than twice the minimum time.

Harmonic oscillator. The quasi time-optimal control law (11), (8b) is globally stabilizing for the system $\dot{x}_1 = x_2$, $\dot{x}_2 = -\omega^2 x_1 - \text{sat}(u)$. For small ε and large k, the time-optimal response is almost recovered, as shown in Fig. 4.

VI. Proof of Theorem 1

The main steps of the proof are the following. We first show in Section VI-A that, for each initial conditions there exists a compact forward invariant set $\mathcal{I}(x_0)$ containing the origin where the trajectory is confined. Then we show in Section VI-B that the origin is the only equilibrium point in the set $\mathcal{I}(x_0)$. Then, to complete the proof of item 1, in Section VI-C we show that there doesn't exist any periodic orbit in the set $\mathcal{I}(x_0)$ so that, by [14, Theorem 18.1, page 66] (following a Bendixson-like approach), all trajectories necessarily converge to the origin. Finally, in Section VI-D we prove item 2 of the theorem.

A. *Existence of the forward invariant set $\mathcal{I}(x_0)$*

Consider the locally Lipschitz Lyapunov function $V = a_1\frac{x_1^2}{2} + \frac{x_2^2}{2} + M|x_1|$. Its generalized gradient [7] in $x_1 = 0$ is the set $\nabla V_0(x_2) = \left\{\left[(1-2\alpha)M \quad x_2\right],\ \alpha \in [0,1]\right\}$, and its generalized derivative along the system dynamics is

$$\dot{V} \in -a_2 x_2^2 - x_2(\text{sat}(k\beta(x)) - M\text{Sgn}\,(x_1)), \tag{14}$$

where the set valued function $\text{Sgn}\,(\cdot)$ is such that $\text{Sgn}\,(x_1) = \text{sgn}\,(x_1)$ if $x_1 \neq 0$, and $\text{Sgn}\,(x_1) = [-1, +1]$ if $x_1 = 0$.

Since k satisfies (3), then $\exists \bar{s} > 0 : k = \frac{M}{\eta(\bar{s})} > \inf_{s>0} \frac{M}{\eta(s)}$. Therefore, by item 1 of Assumption 2 and since $\eta(\cdot) \in \mathcal{K}$,

$$|k\beta(x)| \geq \left|\frac{M}{\eta(\bar{s})}\,\eta(|x|)\right| \geq M, \quad \forall |x| \geq \bar{s}, x_1x_2 \geq 0. \tag{15}$$

Therefore, the following bounded set

$$\mathcal{W} := \{x \,:\, |x| \leq \bar{s}\} \cap \{x \,:\, x_1x_2 > 0\} \subset \{x \,:\, |x| \leq \bar{s}\}$$

is characterized by the fact that any x outside $\mathcal{W}$ and in the (closed) first and third quadrant will lead to (by (15)) $|k\beta(x)| \geq M$, namely will cause the plant input u to saturate.

Consider now the maximum of $V(\cdot)$ in $\mathcal{W}$ and at x_0, namely $\overline{v}(x_0) := \max\{V(x) : x \in \mathcal{W} \cup \{x_0\}\}$, and let $\mathcal{I}$ be the set:

$$\mathcal{I}(x_0) := \{x \,:\, V(x) \leq \overline{v}(x_0)\} \supset \mathcal{W}.$$

$\mathcal{I}$ is compact by radial unboundedness of $V(\cdot)$. By definition of $\mathcal{I}(x_0)$ and item 1 of Assumption 2, the input u is saturated for all $x \in \{x : x_1x_2 \geq 0\}$ (the closed first and third quadrants) intersection with $\overline{\mathcal{I}(x_0)^c}$ (the closed complement of $\mathcal{I}(x_0)$).

Then the following reasonings prove that $\dot{V} \leq 0$ for all $x \in \overline{\mathcal{I}(x_0)^c}$, i.e. $\mathcal{I}(x_0)$ is a forward invariant set where the trajectory is confined (because $x_0 \in \mathcal{I}(x_0)$ by definition):

• **(2nd and 4th quadrants)** if $x_1x_2 \leq 0$, then $k\beta(x)$ could be any value, therefore we only know that $\text{sat}(k\beta(x)) \in [-M, M]$. Based on this, (14) yields:

1) if $x_1 > 0$, $x_2 \leq 0$, then $\dot{V} \leq -x_2(\text{sat}(k\beta(x)) - M) = -|x_2||\text{sat}(k\beta(x)) - M| \leq 0$ (both terms are negative);

2) if $x_1 < 0$, $x_2 \geq 0$, then $\dot{V} \leq -x_2(\text{sat}(k\beta(x)) + M) = -|x_2||\text{sat}(k\beta(x)) + M| \leq 0$ (both terms are positive);

3) if $x_1 = 0$, then, by definition of $\mathcal{I}(x_0)$, $\text{sat}(k\beta(x)) \in \{-M, M\}$ for all $x \in \overline{\mathcal{I}(x_0)^c} \cap \{x_1 = 0\}$. Moreover, also by item 1 of Assumption 2, $\text{sat}(k\beta(x)) = M$ if $x_2 > 0$ and $\text{sat}(k\beta(x)) = -M$ if $x_2 < 0$. Hence $\dot{V} \leq 0$ since

$$\dot{V} \leq \max_{\alpha\in[0,1]} x_2[(1-2\alpha)M - \text{sat}(k\beta(x))] \leq |x_2|M + |x_2|(-M).$$

• **(1st and 3rd quadrants)** if $x_1x_2 > 0$ and $x \in \overline{\mathcal{I}(x_0)^c} \cap \{x_1 = 0\}$, once again $\text{sat}(k\beta(x)) = M\text{sgn}\,(x_2)$ by definition of $\mathcal{I}(x_0)$ and by item 1 of Assumption 2. Then (14) yields $\dot{V} \leq -x_2(\text{sat}(k\beta(x)) - M) = 0$.

B. *Uniqueness of the equilibrium point*

Since $\dot{x}_1 = x_2$ any equilibrium has to be on $x_2 = 0$. By item 1 of Assumption 2 $\text{sat}(k\beta(x)) = 0$ on the x_1 axis if and only if $x_1 = 0$, and has the same sign as x_1 otherwise. Hence on $\{x_2 = 0\}$, $\dot{x}_2 = -a_1x_1 - \text{sat}(k\beta(x)) \neq 0$ if $x_1 \neq 0$. This proves that $x = 0$ is the only equilibrium point.

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C. Proof of convergence using Bendixson's criterion

Since the only equilibrium point is the origin, any trajectory not converging to that point must converge to a periodic orbit contained in $\mathcal{I}(x_0)$. Moreover, any such hypothetical periodic orbit must surround the origin because:

- *no such orbit can happen in a single quadrant*, since the periodicity of the orbit contradicts the property that in each quadrant $\dot{x}_1 = x_2$ is monotone;
- *the trajectory phase is decreasing on the coordinate axes*, since writing the dynamics in polar coordinates [17, §10.5] $|x|^2\dot{\phi} = x_2^2 - x_1(-a_1x_1 - a_2x_2 - \text{sat}(k\beta(x)))$, implying $\dot{\phi} > 0$ on $x_1 = 0$ and on $x_2 = 0$ (by the property of $\beta(\cdot)$ on $x_2 = 0$).

Now, reasoning like in the proof of Bendixson criterion (see, e.g., [17, Lemma 2.2]), suppose by contradiction that there exists a periodic orbit γ including the origin and let S be the region surrounded by γ. Then, denoting the right hand side of the closed-loop system as $f_1(x) := x_2$, $f_2(x) := -a_1x_1 - a_2x_2 - \text{sat}_M(k\beta(x))$, it must necessarily hold that $\int_\gamma f_2(x)dx_1 - f_1(x)dx_2 = 0$, i.e. by Green's theorem,

$$\int\int_S (\partial_{x_1} f_1 + \partial_{x_2} f_2)\, dx_1 dx_2 = 0. \tag{16}$$

However, this cannot happen since, as shown below, item 2 of Assumption 2 implies that any such integral is strictly negative.

To bound the integrand in (16), note that $\text{sat}(\cdot)$ and $\beta(\cdot)$ are locally Lipschitz, hence almost everywhere it holds that:

$$\partial_{x_1} f_1 = 0, \quad \partial_{x_2} f_2 = \begin{cases} -a_2, & \text{if } |k\beta(x)| \geq M, \\ -a_2 - k\partial_{x_2}\beta(x), & \text{if } |k\beta(x)| < M, \end{cases}$$

therefore, for almost all x, $\partial_{x_1} f_1 + \partial_{x_2} f_2 \leq -a_2 \leq 0$. Moreover, since the intersection of S with the set $\mathcal{A}$ (introduced in item 2 of Assumption 2) has a positive measure, then $\partial_{x_1} f_1 + \partial_{x_2} f_2 \leq 0$ and item 2 of Assumption 2, together $a_2 \geq 0$ (by Assumption 1) imply $\int\int_S (\partial_{x_1} f_1 + \partial_{x_2} f_2)\, dx_1 dx_2 < 0$, which contradicts (16). Therefore, no periodic orbit γ exists. Since no periodic trajectory exists in $\mathcal{I}(x_0)$ and the origin is the only equilibrium point, by [14, Theorem 18.1, page 66] (following a Bendixson-like approach), the trajectory necessarily converges to the origin.

D. Proof of item 2 of the theorem

The two conditions at item 2 correspond to requiring strict positiveness of the coefficients of the characteristic polynomial of the linearized system around the origin. Therefore it is straightforward to conclude local exponential stability (LES) of the origin from those conditions. This, together with the global convergence property proven above is sufficient (see, e.g., [29] or [18, pp. 68-72]) to prove global asymptotic and local exponential stability (GAS+LES).

VII. Conclusions

In this paper we have proposed a general approach to the global stabilization of ANCBI linear planar systems subject to input saturation. Our approach allows to achieve almost time (and/or fuel) optimal robust stabilization with a locally Lipschitz state feedback law. Simulation results illustrate the effectiveness of the proposed strategy.

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