Three Dimensional Bin Packing Problem applied to air cargo

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Abstract : *Deciding whether a set of three dimensional boxes can be packed into a container is a NP-hard problem. Mathematical models have been developed, however, only few studies take into account constraints encountered in real-world applications such as the stability or the fragility of the cargo. Moreover, despite the importance of this issue in air transport, the literature is almost silent on constraints related to the distribution of the weight inside a container. This paper is concerned with the formulation of the three dimensional palletization which includes the main constraints met in the air cargo industry.*

Keywords : *Bin-packing, cargo aircraft, unit load devices*

1 Introduction

Nowadays, packing boxes into containers is a daily process in many fields such as truck or air transport. This process has to be conducted as fast and profitable as possible. Indeed, it is important to pack a maximum number of boxes into a minimum number of containers such that the costs can be reduced. This minimization problem is called the *bin packing problem* (BPP) or *container loading problem*. Bin packing doesn't only concern the transport; it can be applied in different fields. For instance, trying to encode some given electronical data on a minimum number of DVDs is also considered as bin packing.

There exist some variants of the BPP: it can be considered in one, two or three dimensions and with one or several containers. If there are several containers, they can have the same or different shapes (*Multi-Container Loading Problem*) and so do the boxes. The *knapsack problem* is a related problem: only one container is given and each box is associated to a weight and a value. The aim is to load the boxes into the container maximizing the value of the loaded set without exceeding a given maximal weight. [5] developed a more complete classification of the cutting and packing problem. Some specific constraints inherent to the intended application can be added to the initial problem. For example, the cost of booking the containers could be taken into account ([2, 14]).

A lot of articles developed some models for the BPP on one and two dimensions while the three dimensional BPP (3D-BPP) is a more recent subject, the first models appeared less than twenty years ago (see e.g. [3, 10, 13]). However, the application to the air transport still needs to be studied because of its particular constraints. Yet, since BPP is a NP-hard problem, it is unlikely that a polynomial algorithm which optimally solves all instances of the BPP could be found. For this reason, some heuristics methods are developed to try to solve the BPP (see e.g. [2, 8, 15]).

In this paper, 3D-BPP is considered in the particular case of air cargo. In this specific situation, containers are called *unit load devices* (ULD). A ULD is an assembly of components consisting of a container or of a pallet covered with a net, whose purpose is to provide standardized size units for individual pieces of baggage or cargo, and to allow for rapid loading and unloading ([9]). After being filled in, these ULDs will be loaded into a compartmentalised cargo aircraft with some technical and safety constraints (see e.g. [1, 9, 11, 6, 12]). One of these constraints requires that the centre of gravity of the loaded plane should be as close as possible to a recommended position determined by safety and fuel economy considerations. In this purpose, the weight of each ULD is supposed to be uniformly distributed or, at least, well-balanced. This assumption will be integrated as a constraint in the following model ([4, 7]). Note that the ULDs may differ in size, shape and maximum weight. The most common ULDs are illustrated in Figure 1. Even if the shape of ULDs isn't always a rectangular parallelepiped, it will be treated as such in the hereinafter.

2 Model

The description of the problem is the following one. The first step is to consider a set of n rectangular boxes of dimensions $l_i \times w_i \times h_i$ and weight m_i ($i \in \{1, ..., n\}$) that has to be packed into identical containers of dimensions $L \times W \times H$ and maximal capacity C. Based on the classification of [5], this problem is 3/V/I/M. That is, the problem is considered in three dimensions (3) in which all boxes are assigned to a selection of containers (V), whose shapes are identical (I) and there are many boxes of many different shapes (M). In this paper, we propose an original model whose goal is to pack the n boxes into ULDs such that the unused space is minimized

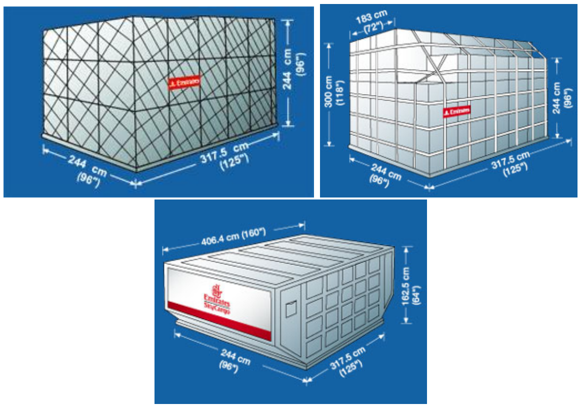

Figure 1: Most common ULDs: Q6 on the top left, Q7 on the top right, LD on the bottom.

by respecting some basic and specific constraints. Since all the ULDs are identical, it is equivalent to minimize their number.

2.1 Parameters

The following data are known: the number of boxes to be packed, the dimensions and weight of each box and the dimensions and maximum gross weight of all the containers. These parameters are denoted in the model by

n	Total number of boxes to be packed
m	Total number of available ULDs
$l_i \times w_i \times h_i$	Length $\times$ width $\times$ height of box i
p_i	Weight of box i
	$0 \leq i \leq n$
$L \times W \times H$	Length $\times$ width $\times$ height of containers
C	Maximum gross weight of containers
M	A large number.

All the containers are assumed identical and available in sufficient quantities.

2.2 Variables

Here are the different variables used in the model.

$$p_{ij} = \begin{cases} 1 & \text{if box } i \text{ is in container } j \\ 0 & \text{otherwise.} \end{cases}$$

$$u_j = \begin{cases} 1 & \text{if container } j \text{ is used} \\ 0 & \text{otherwise.} \end{cases}$$

(x_i, y_i, z_i) Location of the front left bottom corner of box i

(x'_i, y'_i, z'_i) Location of the rear right top corner of box i

$$x^b_{ki} = \begin{cases} 1 & \text{if } x'_k < x_i \\ 0 & \text{otherwise.} \end{cases}$$

$$x^a_{ki} = \begin{cases} 1 & \text{if } x_k > x'_i \\ 0 & \text{otherwise.} \end{cases}$$

$$y^b_{ki} = \begin{cases} 1 & \text{if } y'_k < y_i \\ 0 & \text{otherwise.} \end{cases}$$

$$y^a_{ki} = \begin{cases} 1 & \text{if } y_k > y'_i \\ 0 & \text{otherwise.} \end{cases}$$

$$z^b_{ki} = \begin{cases} 1 & \text{if } z'_k < z_i \\ 0 & \text{otherwise.} \end{cases}$$

$$z^a_{ki} = \begin{cases} 1 & \text{if } z_k > z'_i \\ 0 & \text{otherwise.} \end{cases}$$

r^i_{pq} Binary variables used to describe the orientation of box i into a container

$\forall i, k \in \{1, ..., n\}, \forall j \in \{1, ..., m\}, \forall p, q \in \{1, 2, 3\}$.

The variable x^b_{ki} (resp. x^a_{ki}, y^b_{ki}, y^a_{ki}, z^b_{ki}, z^a_{ki}) is equal to 1 if the box i is on the left (resp. right, behind, in front of, above, below) of box k.

Without loss of its generality, the axes of the coordinate system are assumed to be placed so that the length L (resp. width W, height H) of the container lies on the x-axis (resp. y-axis, z-axis). About the boxes, when the box i is initially given, the measure of the edge along the x-axis (resp. the y-axis, the z-axis) is defined as the length l_i (resp. the width w_i, the height h_i). In other words, by convention, the equation

$$(x'_i - x_i, y'_i - y_i, z'_i - z_i) = (l_i, w_i, h_i) \tag{1}$$

is always true at the first step.

The representation of some of these parameters and variables, after optimization and some rotations, is given in Figure 2.

In Figure 2, one has

$$\begin{array}{ll} x^b_{ji} = 1 & x^b_{ij} = 0 \\ x^a_{ji} = 0 & x^a_{ij} = 1 \\ y^b_{ji} = 0 & y^b_{ij} = 0 \\ y^a_{ji} = 0 & y^a_{ij} = 0 \\ z^b_{ji} = 0 & z^b_{ij} = 0 \\ z^a_{ji} = 0 & z^a_{ij} = 0. \end{array}$$

2.3 Objective function

The objective function consists in minimizing the unused volume of the ULDs

$$\sum_{j=1}^{m} u_j (L \cdot W \cdot H) - \sum_{i=1}^{n} l_i \cdot w_i \cdot h_i. \tag{2}$$

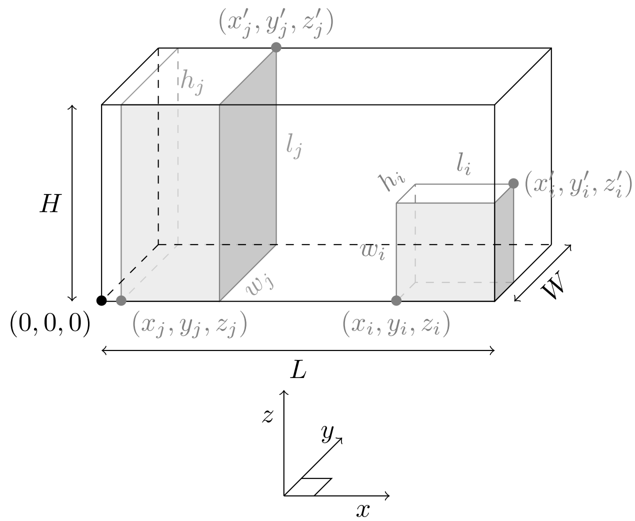

Figure 2: Representation of some parameters and variables: the container is in black, the boxes i and j are in gray, the coordinate system is below.

As said before, since the ULDs are supposed to have the same shape, (2) is equivalent to minimize the number of ULDs. Mathematically, since l_i, w_i, h_i are parameters initially determined, the term $\sum_{i=1}^{n} l_i \cdot w_i \cdot h_i$ is a constant, as well as $L \cdot W \cdot H$. So, the objective function consists in minimizing the number of ULDs

$$\sum_{j=1}^{m} u_j. \tag{3}$$

However, if there were different sizes or shapes of ULDs, it wouldn't be equivalent. Indeed, the number of ULDs could be minimized without minimizing the unused volume. For instance, a big ULD could be selected such that all the boxes are packed into it, but it is too big and there is a lot of unused volume. Maybe, two smaller ULDs should have been selected so that there is less unused volume. The procedure should be different. However, in this paper, the ULDs are identical. Thus, this situation doesn't have to be considered.

2.4 Constraints

In this subsection, a difference between basic and specific constraints is made. First, the 3D-BPP has some general requirements such as the non overlapping of the boxes. Indeed, it seems logical that two different boxes may not occupy a same part of the container. Second, some specific constraints could appear when handling real shipments: uniform distribution of the weight, rotations, stability, fragility, ...

2.4.1 Basic constraints

Here are the basic constraints of the 3D-BPP.

$$p_{ij} \leq u_j \quad \forall i, j \tag{4}$$

$$\sum_{j=1}^{m} p_{ij} = 1 \quad \forall i \tag{5}$$

$$\sum_{i=1}^{n} p_{ij}\, m_i \leq C\, u_j \quad \forall j \tag{6}$$

$$x'_i \leq L \quad \forall i \tag{7}$$

$$y'_i \leq H \quad \forall i \tag{8}$$

$$z'_i \leq W \quad \forall i \tag{9}$$

$$x'_i - x_i = r^i_{11}\, l_i + r^i_{12}\, w_i + r^i_{13}\, h_i \quad \forall i \tag{10}$$

$$y'_i - y_i = r^i_{21}\, l_i + r^i_{22}\, w_i + r^i_{23}\, h_i \quad \forall i \tag{11}$$

$$z'_i - z_i = r^i_{31}\, l_i + r^i_{32}\, w_i + r^i_{33}\, h_i \quad \forall i \tag{12}$$

$$\sum_{p} r^i_{pq} = 1 \quad \forall i, q \tag{13}$$

$$\sum_{q} r^i_{pq} = 1 \quad \forall i, p \tag{14}$$

$$x_i - x'_k + (1 - x^b_{ki})M + (2 - (p_{ij} + p_{kj}))M \geq 0 \quad \forall i, j, k \tag{15}$$

$$x_k - x'_i + (1 - x^a_{ki})M + (2 - (p_{ij} + p_{kj}))M \geq 0 \quad \forall i, j, k \tag{16}$$

$$y_i - y'_k + (1 - y^b_{ki})M + (2 - (p_{ij} + p_{kj}))M \geq 0 \quad \forall i, j, k \tag{17}$$

$$y_k - y'_i + (1 - y^a_{ki})M + (2 - (p_{ij} + p_{kj}))M \geq 0 \quad \forall i, j, k \tag{18}$$

$$z_i - z'_k + (1 - z^b_{ki})M + (2 - (p_{ij} + p_{kj}))M \geq 0 \quad \forall i, j, k \tag{19}$$

$$z_k - z'_i + (1 - z^a_{ki})M + (2 - (p_{ij} + p_{kj}))M \geq 0 \quad \forall i, j, k \tag{20}$$

$$x^b_{ki} + x^a_{ki} + y^b_{ki} + y^a_{ki} + z^b_{ki} + z^a_{ki} > 0 \quad \forall i, k \tag{21}$$

$$i, k \in \{1, ..., n\}, j \in \{1, ..., m\}, p, q \in \{1, 2, 3\}.$$

The constraint (4) ensures that a box is assigned to a container only if this one is used. (Conversely, a container is considered as used if any box is assigned to it). The constraint (5) verifies that a box is allocated to exactly one container. The maximum capacity cannot be exceeded which is checked by constraint (6). Constraints (7)-(9) ensures that the boxes don't exceed the container. The constraints (10)-(14) describe that the boxes can rotate orthogonally in the container. In other words, the edges of a box are either parallel to or perpendicular to the edges of the container. These constraints could be expressed in a matrix form

$$\begin{pmatrix} x_i' - x_i \\ y_i' - y_i \\ z_i' - z_i \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11}^i & r_{12}^i & r_{13}^i \\ r_{21}^i & r_{22}^i & r_{23}^i \\ r_{31}^i & r_{32}^i & r_{33}^i \end{pmatrix}}_{R^i} \cdot \begin{pmatrix} l_i \\ w_i \\ h_i \end{pmatrix}$$

where R^i is a permutation matrix. That is, each column and row of R^i contain only one element equal to one, the other ones are set to zero. This is represented in the constraints (13)-(14). If we initially set the length l_i, the width w_i and the height h_i of box i respectively along the axes x, y, z as in (1), we get:

$$R^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the boxes i and j in Figure 2, one has

$$R^i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$R^j = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that (10)-(12) implies $x_i < x_i'$, $y_i < y_i'$, $z_i < z_i'$. The non overlapping of the boxes is handled by constraints (15)-(21). The term $(2-(p_{ij}+p_{kj}))M$ ensures that the condition of non overlapping must be satisfied only if the two boxes are in the same container. The constraints (15)-(20) link the variables x_i, x_i', y_i, y_i', z_i, z_i', x_k, x_k', y_k, y_k', z_k, z_k' with the variables x_{ki}^b, x_{ki}^a, y_{ki}^b, y_{ki}^a, z_{ki}^b, z_{ki}^a; i.e. the position of each side of a box i with respect to the sides of another box k. The constraint (21) is the constraint of non overlapping. The parameter M is chosen to be set at $L + W + H$ so that it has a value greater than any other values considered in the model.

2.4.2 Specific constraints

As said before, the application of the 3D-BPP to the real world applications implies some specific constraints. Here, four types of constraints are studied. The first constraint considered the boxes that may not rotate in all directions. The second one, specially important in air cargo, requires that the weight must be evenly distributed in the ULD assuming that it is the case in each box. Moreover, since the boxes are assumed to be stackable, the two last types naturally appear: the cargo stability and the load bearing constraints.

Some boxes could not be allowed to rotate in all directions because of their content; e.g. some products may not turn upside-down. In this purpose, some new parameters are introduced:

$$vert\_l_i = \begin{cases} 1 & \text{if the length of box } i \text{ could be} \\ & \text{ in a vertical position} \\ 0 & \text{otherwise} \end{cases}$$
$$vert\_w_i = \begin{cases} 1 & \text{if the width of box } i \text{ could be} \\ & \text{ in a vertical position} \\ 0 & \text{otherwise} \end{cases}$$
$$vert\_h_i = \begin{cases} 1 & \text{if the height of box } i \text{ could be} \\ & \text{ in a vertical position} \\ 0 & \text{otherwise.} \end{cases}$$

Note that at least one of these parameters should equal one, otherwise there is no possible configuration. If all the parameters are set to the value 1, then six configurations exist. In fact, six different configurations of the space occupied by the box exist. More precisely, we just consider the 90°-rotation since a 180°-rotation does not change the space used by the box. So, the space occupied by the box is described more than the box itself. For each parameter equal to 0, two configurations are not possible anymore. For example, consider the case where the parameter $vert\_l_i$ equals 0. Then the four allowed configurations are as in Figure 3.

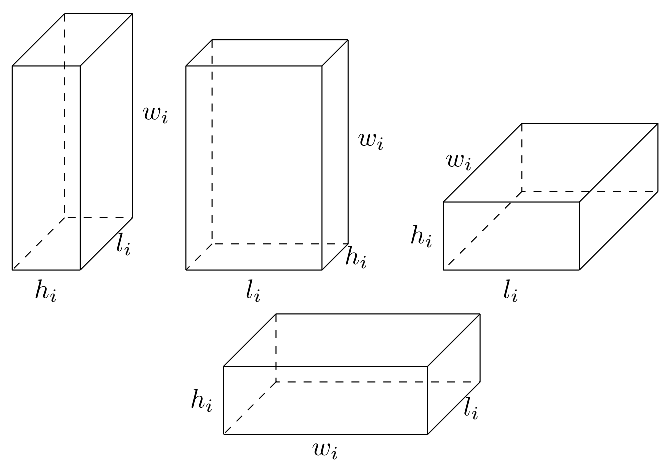

Figure 3: Possible configurations for the box i if the length of the box could not be along the z-axis

In the same way, if two parameters equal 0, only two configurations of the box remains possible. In fact, these parameters control the third row of the matrix R^i.

As defined above, without any constraint, one has this system

$$\begin{pmatrix} x_i' - x_i \\ y_i' - y_i \\ z_i' - z_i \end{pmatrix} = \begin{pmatrix} r_{11}^i & r_{12}^i & r_{13}^i \\ r_{21}^i & r_{22}^i & r_{23}^i \\ r_{31}^i & r_{32}^i & r_{33}^i \end{pmatrix} \begin{pmatrix} l_i \\ w_i \\ h_i \end{pmatrix}.$$

If the parameter $vert\_l_i$ (resp. $vert\_w_i$, $vert\_h_i$) equals 0, then the third row will be $(0 \quad r_{32} \quad r_{33})$ (resp.$(r_{31} \quad 0 \quad r_{33})$, $(r_{31} \quad r_{32} \quad 0)$). In the case where two parameters equal 0, two 0 are on the same row and thus the last parameter has to be 1 to satisfy (13)- (14).

Let us now consider three other constraints.

In order to load an aircraft (or a truck), an even distribution of the weight inside the ULDs is a desirable property and simplifies the modelisation of the problem. In this purpose, we make the hypothesis that the weight of each box is uniformly distributed. This constraint will be added in the model to ensure that the ULD is balanced.

Moreover, the cargo stability involves the vertical (or static) and the horizontal (or dynamic) stability. About the vertical stability, the bottom face of each box has to be supported either by the top face of other boxes or by the container floor. Thus, the boxes are not displaced with respect to z-axis. This constraint is also called *static stability* because it deals with containers which don't move. The vertical constraint excludes floating boxes. About the horizontal stability, it deals with moving container. It refers to the capacity of the box to withstand the inertia of its own body when moving. The boxes remains in their positions with respect to x and y axes, hence the name of *horizontal stability*.

To avoid loading patterns where boxes are floating in midair (vertical stability not respected), the bottom face of each box has to be supported either by the top face of other boxes or by the container floor. Thus, for a box i, either z_i equals 0 or there exists a box k such that $z_i = z'_k$ and there is an overlap with respect to xy-plane. We have to take into account that the overlap on the xy-plane is large enough to support the box.

Load bearing strength refers to the maximum number of boxes that can be stacked one above each other. More generally, it refers to the maximum pressure that can be applied over the top face of a box avoiding damaging the box (7). For instance, a box is said fragile if no box can be placed above its top face. This constraint is quite important in practice because it avoids damaging products contained in a fragile box.

Other types of constraints may exist such as the *guillotine* constraint, which imposes that the patterns be such that items can be obtained by sequential face-to-face cuts parallel to the faces of the container (10). This is not considered here. Another well-known set of constraints is the multi-drop one which refers to cases where boxes that are delivered to the same destination must be placed close to each other in the container. The loading patterns must take into account the delivery route of the vehicle and the order in which the boxes are unloaded.

3 Implementation

The model presented is already implemented in Java with the professional library IBM ILOG CPLEX 12. Since the model is a mixed integer linear program, we have used the classical branch-and-cut CPLEX solver with the default parameters. But, due to its complexity, this problem opens the way to heuristics.

4 Conclusion

The bin packing problem is a current problem encountered in transport. This paper presents a mixed integer linear programming model for the particular case of the three dimensional bin packing problem applied to the air transport. This specific application involves new constraints such as the stability and the fragility of the cargo. The model gives a minimum number of ULDs to use and it proposes some actual loading patterns.

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