Equivariant quantization in supergeometry

Fabian Radoux

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Introduction

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- Quantum observables: operators on a Hilbert space $\mathcal{H}$. Example: $\mathcal{H} = L^2(T^*M)$. 

Dirac problem: find a bijection $Q: \mathcal{C}^\infty(T^*M) \to L(\mathcal{H})$. First answer to the Dirac problem: prequantization $Q$. Reduction of $\mathcal{H}$, $L^2(T^*M)$ is replaced by $L^2(M)$. The observable $f$ is quantizable if $Q(f)$ preserves $L^2(M)$. The set of quantizable observables: $\text{Pol} \leq 1(T^*M)$. 

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- The set of quantizable observables: \(\text{Pol}_{\leq 1}(T^*M)\).
Geometric quantization $Q_G$: $Q_G = Q|_{\text{Pol}_{\leq 1}(T^*M)}$,

$$Q_G(X^i(x)p_i + A(x)) = \frac{\hbar}{i} X^i(x) \partial_i + A(x).$$
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Is this prolongation unique?
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- Is it possible to extend the geometric quantization to $\text{Pol}(T^*M) \cong S(M)$?
- Is this prolongation unique?
- Is it possible to reestablish the uniqueness?
There is no natural quantization: there is no linear bijection $Q : S(M) \rightarrow D(M)$ such that

$$\Phi^*(Q(S)) = Q(\Phi^*S)$$

for all local diffeomorphism $\Phi$. 
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For example: $Q_{\text{aff}}$ defined by

$$Q_{\text{aff}}(S^{i_1 \cdots i_k} \partial_{i_1} \vee \cdots \vee \partial_{i_k}) = S^{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k}$$

is not well-defined: if $J$ denotes the Jacobian of the change of variables $\bar{x}(x)$,
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Equivariant quantization: action of a Lie group $G$ on $M$: 
$\Phi : G \times M \to M$. 

Projective case (P. Lecomte, V. Ovsienko):
$PGL(m+1, \mathbb{R})$ acts on $\mathbb{R}P^m$ locally diffeomorphic to $\mathbb{R}^m$.
$X \in sl(m+1, \mathbb{R}) \mapsto \text{vector field on } \mathbb{R}^m$.

$\exists Q : L_X Q(S) = Q(L_X S)$ for all $X \in sl(m+1, \mathbb{R})$. 

Idea: to take $G$ sufficiently small to have a quantization and sufficiently big to have the uniqueness.
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Equivariant quantization $Q$: linear bijection $Q$: 
$S(M) \rightarrow D(M)$ s.t. $\sigma(Q(S)) = S$ and s.t. 
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- **Equivariant quantization**: action of a Lie group \( G \) on \( M \):
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  - \( X \in \mathfrak{sl}(m+1, \mathbb{R}) \leftrightarrow X^* \) vector field on \( \mathbb{R}^m \).
  - \( \exists Q : L_X Q(S) = Q(L_X S) \quad \forall X \in \mathfrak{sl}(m+1, \mathbb{R}). \)
- Conformal case (C. Duval, P. Lecomte, V. Ovsienko):
- $SO(p + 1, q + 1)$ acts on $S^p \times S^q$. 
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- Casimir operator method:
  - $l$: Semi-simple Lie algebra endowed with a non degenerate Killing form $K$. 
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  - $(V, \beta)$: representation of $\mathfrak{l}$.
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Casimir operator method:

- $(V, \beta)$: representation of $L$.
- $(u_i : i \leq n)$: basis of $L$; $(u'_i : i \leq n)$: Killing-dual basis $(K(u_i, u'_j) = \delta_{i,j})$. 
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- $(V, \beta)$: representation of $l$.
- $(u_i : i \leq n)$: basis of $l$; $(u'_i : i \leq n)$: Killing-dual basis ($K(u_i, u'_j) = \delta_{i,j}$).

Casimir operator corresponding to $(V, \beta)$:

$$\sum_{i=1}^{n} \beta(u'_i)\beta(u_i).$$
(\mathcal{S}(\mathbb{R}^m), L) and (\mathcal{D}(\mathbb{R}^m), \mathcal{L}) are representations of \mathfrak{g}.
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\(C\) and \(\check{C}\): Casimir operators of \(\mathfrak{g}\) on \(\mathcal{S}(M)\) and \(\mathcal{D}(M)\).


- $(S(\mathbb{R}^m), L)$ and $(D(\mathbb{R}^m), \mathcal{L})$ are representations of $\mathfrak{g}$.
- $\mathcal{C}$ and $\mathcal{C}$: Casimir operators of $\mathfrak{g}$ on $S(M)$ and $D(M)$.
- If $\mathcal{C}(S) = \alpha S$ and $\mathcal{L} \circ Q = Q \circ L$, then $\mathcal{C}(Q(S)) = \alpha Q(S)$. 


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\( C \) and \( \mathcal{C} \): Casimir operators of \( \mathfrak{g} \) on \( \mathcal{S}(M) \) and \( \mathcal{D}(M) \).

If \( C(S) = \alpha S \) and \( \mathcal{L} \circ Q = Q \circ L \), then
\[
C(Q(S)) = \alpha Q(S).
\]

In non-critical situations: if \( C(S) = \alpha S \), then \( \exists! \) \( Q(S) \) s.t. \( C(Q(S)) = \alpha Q(S), \sigma(Q(S)) = S \).
(S(\mathbb{R}^m), L) and (D(\mathbb{R}^m), \mathcal{L}) are representations of \mathfrak{g}.

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In these conditions: \(\mathcal{L}(Q(S)) = Q(L(S))\) because:
\begin{itemize}
\item \((S(\mathbb{R}^m), L)\) and \((D(\mathbb{R}^m), \mathcal{L})\) are representations of \(\mathfrak{g}\).
\item \(C\) and \(\mathcal{C}\): Casimir operators of \(\mathfrak{g}\) on \(S(M)\) and \(D(M)\).
\item If \(C(S) = \alpha S\) and \(L \circ Q = Q \circ L\), then \(C(Q(S)) = \alpha Q(S)\).
\item In non-critical situations: if \(C(S) = \alpha S\), then \(\exists!\ Q(S)\) s.t. \(C(Q(S)) = \alpha Q(S), \sigma(Q(S)) = S\).
\item In these conditions: \(L(Q(S)) = Q(L(S))\) because:
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\end{itemize}
(\(S(\mathbb{R}^m), L\)) and (\(D(\mathbb{R}^m), L\)) are representations of \(\mathfrak{g}\).

- \(C\) and \(\mathcal{C}\): Casimir operators of \(\mathfrak{g}\) on \(S(M)\) and \(D(M)\).

- If \(C(S) = \alpha S\) and \(L \circ Q = Q \circ L\), then \(C(Q(S)) = \alpha Q(S)\).

- In non-critical situations: if \(C(S) = \alpha S\), then \(\exists!\ Q(S)\) s.t. \(C(Q(S)) = \alpha Q(S),\ \sigma(Q(S)) = S\).

- In these conditions: \(L(Q(S)) = Q(L(S))\) because:
  - \(\sigma(L(Q(S))) = \sigma(Q(L(S))) = L(S)\);
  - \(C(Q(L(S))) = \alpha Q(L(S)),\ \mathcal{C}(L(Q(S))) = \alpha \mathcal{L}(Q(S))\).
Conjecture (P. Lecomte): $Q(\nabla) : S(M) \rightarrow \mathcal{D}(M)$:
Conjecture (P. Lecomte): \( Q(\nabla) : S(M) \to \mathcal{D}(M) \):

1. Natural: \( \Phi^*(Q(\nabla)(S)) = Q(\Phi^*\nabla)(\Phi^*S) \) for all local diffeomorphism \( \Phi \)
Conjecture (P. Lecomte): \( Q(\nabla) : S(M) \rightarrow \mathcal{D}(M) : \)

1. **Natural:** \( \Phi^*(Q(\nabla)(S)) = Q(\Phi^*\nabla)(\Phi^*S) \) for all local diffeomorphism \( \Phi \)

2. **Projectively invariant:** \( Q(\nabla) = Q(\nabla') \) if \( \nabla' = \nabla + \alpha \lor \text{id} \)
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$\varphi_t^* Q(\nabla_0)(S) = Q(\varphi_t^*\nabla_0)(\varphi_t^*S)$, $\nabla_0$ flat connection of $\mathbb{R}^m$, $\varphi_t$ flow of $X \in \mathfrak{sl}(m + 1, \mathbb{R})$
- **Conjecture (P. Lecomte)**: \( Q(\nabla) : \mathcal{S}(\mathcal{M}) \to \mathcal{D}(\mathcal{M}):\)
  1. **Natural**: \( \Phi^*(Q(\nabla)(S)) = Q(\Phi^*\nabla)(\Phi^*S) \) for all local diffeomorphism \( \Phi \)
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- \( \varphi_t^* Q(\nabla_0)(S) = Q(\varphi_t^*\nabla_0)(\varphi_t^*S) \), \( \nabla_0 \) flat connection of \( \mathbb{R}^m \), \( \varphi_t \) flow of \( X \in \mathfrak{sl}(m+1, \mathbb{R}) \)

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1. Natural: \( \Phi^*(Q(\nabla)(S)) = Q(\Phi^*\nabla)(\Phi^*S) \) for all local diffeomorphism \( \Phi \)

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- \( \varphi^*_t Q(\nabla_0)(S) = Q(\varphi^*_t \nabla_0)(\varphi^*_t S) \), \( \nabla_0 \) flat connection of \( \mathbb{R}^m \), \( \varphi_t \) flow of \( X \in \mathfrak{sl}(m+1, \mathbb{R}) \)

- \( \varphi^*_t Q(\nabla_0)(S) = Q(\nabla_0)(\varphi^*_t S) \) because \( \varphi^*_t \nabla_0 \sim \nabla_0 \) and \( Q \) projectively invariant

- \( L_X Q(\nabla_0)(S) = Q(\nabla_0)(L_X S) \) for all \( X \in \mathfrak{sl}(m+1, \mathbb{R}) \)
Projective case, differential operators acting between densities: M. Bordemann method:
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- $M \mapsto \tilde{M}$: fiber bundle of rank one over $M$ (Thomas fiber bundle)
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- Connection $\nabla$ on $M \mapsto$ Connection $\tilde{\nabla}$ on $\tilde{M}$ associated with $\nabla$ in a natural and projectively invariant way (Thomas connection)
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- $\tilde{Q}(\tilde{\nabla})(\tilde{S})(\tilde{f}) = \tau(\tilde{\nabla})(\tilde{S})(\tilde{f})$ with $\tau$ a canonical natural quantization
Aims of the talk

Show how to superize and to solve in the super setting the following problems:

- Projectively equivariant quantization on $\mathbb{R}^m$
- Conformally equivariant quantization on $\mathbb{R}^m$
- Natural and projectively invariant quantization
Superfunction $f$ on a supermanifold of dimension $(n|m)$:
locally, $f(x^1, \ldots, x^n) = \sum_{I \subseteq \{1, \ldots, m\}} f_I(x^1, \ldots, x^n) \theta^I$, $\theta^i \theta^j = -\theta^j \theta^i$. 
Projectively equivariant quantization on $\mathbb{R}^{n|m}$ (P. Mathonet, R.)

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Locally, a $\lambda$-density is expressed formally as $f |Dx|^\lambda$. Under a change of coordinates $\bar{x}(x)$, $|Dx|^\lambda$ is multiplied by $|\text{Ber}A|^\lambda$, with $A^i_j = \frac{\partial \bar{x}^j}{\partial x^i}$. 
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- Super vector field: superderivation of the superalgebra of superfunctions.

- Locally, a $\lambda$-density is expressed formally as $f|Dx|^\lambda$. Under a change of coordinates $\tilde{x}(x)$, $|Dx|^\lambda$ is multiplied by $|\text{Ber}A|^\lambda$, with $A^i_j = \frac{\partial \tilde{x}^j}{\partial x^i}$. Moreover,

$$L_X(f|Dx|^\lambda) = (X(f) + \lambda \text{div}(X)f)|Dx|^\lambda,$$

where

$$\text{div}(X) = \sum_{i=1}^{n+m} (-1)^{\tilde{y}_i} X_i \partial_{y_i} X^i.$$
Differential operator $D \in \mathcal{D}^k_{\lambda, \mu}$:

$$D = \sum_{|\alpha| \leq k} D_\alpha \partial_{\chi^1}^{\alpha_1} \cdots \partial_{\chi^n}^{\alpha_n} \partial_{\theta^1}^{\alpha_{n+1}} \cdots \partial_{\theta^m}^{\alpha_{n+m}},$$

where $D_\alpha$ are local $\delta$-densities ($\delta = \mu - \lambda$).
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- $\mathcal{L}_X D = \mathcal{L}_X \circ D - (-1)^{\bar{X} \bar{D}} D \circ \mathcal{L}_X$.

- Space of symbols isomorphic to the graded space associated with $\mathcal{D}_{\lambda, \mu}$, isomorphism induced by:

$$\sigma_k : \mathcal{D}^k \rightarrow S^k : D \mapsto \sum_{|\alpha| = k} D_\alpha \otimes \partial_1^{\alpha_1} \vee \ldots \vee \partial_{n+m}^{\alpha_{n+m}}.$$
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- Quantization: linear bijection $Q : S_\delta \to \mathcal{D}_{\lambda,\mu}$ s.t. $\sigma_k(Q(S)) = S$ for all $S \in S^k_\delta$. 

Projective superalgebra of vector fields on $\mathbb{R}^{n|m}$

- $\mathfrak{pgl}(n + 1|m) = \mathfrak{gl}(n + 1|m)/\mathbb{R}\text{Id} \leftrightarrow \text{subalgebra of vector fields over } \mathbb{R}^{n|m}$. 
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\[
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\uparrow i & & \downarrow i^{-1} \\
C^\infty n|m & \xrightarrow{\pi(h_{n+1,m}(A))} & C^\infty n|m
\end{array}
\]
\( \pi \circ h_{n+1,m}(\text{Id}) = 0, \) thus \( \pi \circ h_{n+1,m} \) induces a homomorphism from \( \mathfrak{pgl}(n+1|m) \) to \( \text{Vect}(\mathbb{R}^{n|m}) \).
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\textbf{Projectively equivariant quantization on } \mathbb{R}^{n|m}:

quantization \( Q \) s.t. \( \mathcal{L}_{X_h} \circ Q = Q \circ \mathcal{L}_{X_h} \) for every \( h \in \mathfrak{pgl}(n+1|m) \).
Construction of the quantization

- Casimir operators:
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\( \mathfrak{l} \): Lie superalgebra endowed with a nondegenerate even supersymmetric bilinear form \( K \).
Construction of the quantization

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- \((u_i : i \leq n)\): homogeneous basis of \( \mathfrak{l} \); \((u'_i : i \leq n)\): \( K \)-dual basis \( (K(u_i, u'_j) = \delta_{i,j}) \).
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- **Casimir operator of \((\mathcal{V}, \beta)\):**

\[
\sum_{i=1}^{n} (-1)^{\bar{u}_i} \beta(u_i) \beta(u'_i) = \sum_{i=1}^{n} \beta(u'_i) \beta(u_i).
\]
If $m \neq n + 1$, $\text{pgl}(n + 1|m) \cong \text{sl}(n + 1|m)$. 
If $m \neq n + 1$, \( \mathfrak{pgl}(n + 1|m) \cong \mathfrak{sl}(n + 1|m) \).

Killing form of \( \mathfrak{sl}(n + 1|m) \):

\[
K(A, B) = \text{str}(\text{ad}(A)\text{ad}(B)) = 2(n + 1 - m)\text{str}(AB).
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\( K \) allows to define \( C \) and \( \mathcal{C} \) corresponding resp. to \((S, L)\) and \((\mathcal{D}, \mathcal{L})\).
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The Casimir operator \( C \) of \( \mathfrak{pgl}(n + 1|m) \cong \mathfrak{sl}(n + 1|m) \) on \((S^k_\delta, L)\) is equal to \( \alpha(k, \delta)\text{Id} \).
If $\delta$ is not critical, then there exists a unique projectively equivariant quantization.
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**Proof:**

1. For every $S \in S^k_\delta$, $\exists \hat{S}$ s.t. $C(\hat{S}) = \alpha(k, \delta)\hat{S}$ and s.t. $\sigma(\hat{S}) = S$. 
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Proof:

1. For every $S \in S_{\delta}^{k}$, $\exists! \hat{S}$ s.t. $C(\hat{S}) = \alpha(k, \delta)\hat{S}$ and s.t. $\sigma(\hat{S}) = S$.
2. $Q(S) := \hat{S}$.
3. If $S \in S_{\delta}^{k}$, $Q(L_{Xh}S) = L_{Xh}(Q(S))$ because they are eigenvectors of $C$ of eigenvalue $\alpha(k, \delta)$ and because their symbol is $L_{Xh}S$. 
Divergence operator:

\[
\text{div} : S_\delta^k \to S_\delta^{k-1} : S \mapsto \sum_{j=1}^{n+m} (-1)^{\tilde{y}^j} i(dx^j) \partial_{y^j} S.
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**Theorem**

*If $\delta$ is not critical, then the map $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ defined by*

$$Q(S)(f) = \sum_{r=0}^{k} C_{k,r} Q_{\text{Aff}}(\text{div}^r S)(f), \quad \text{for all } S \in \mathcal{S}_\delta^k$$

*is the unique $\mathfrak{sl}(n+1|m)$-equivariant quantization if*

$$C_{k,r} = \frac{\prod_{j=1}^{r}((n-m+1)\lambda + k-j)}{r! \prod_{j=1}^{r}(n-m+2k-j-(n-m+1)\delta)} \quad \forall r \geq 1.$$
Case $m = n + 1$: $\text{pgl}(n + 1|n + 1)$ not endowed with a non-degenerate bilinear symmetric invariant form.
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Killing form of $\mathfrak{psl}(n + 1|n + 1)$ vanishes, but $K$ defined by

$$K([A], [B]) = \text{str} AB$$

is a nondegenerate invariant supersymmetric even form, we can then apply the Casimir operator method to $\mathfrak{psl}(n + 1|n + 1)$. 
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If $k \neq 1$, $Q$ is given by the same formula as in the case $m \neq n + 1$. 
If $k = 1$,

$$Q_1 : S \mapsto Q(S) = Q_{\text{Aff}}(S + t \text{div}(S))$$

defines a $\mathfrak{psl}(n + 1|n + 1)$-equivariant quantization for every $t \in \mathbb{R}$ (vector fields in $\mathfrak{psl}(n + 1|n + 1)$ are divergence-free).
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\( Q \) does not depend on \( \delta \) and \( \lambda \).
Orthosymplectically equivariant quantizations on $\mathbb{R}^{n|2r}$ (T. Leuther, P. Mathonet, R.)

- $\mathfrak{osp}(p + 1, q + 1|2r)$:

$$\{ A \in \mathfrak{gl}(p+q+2|2r) : \omega(AU, V) + (-1)^{\tilde{A}\tilde{U}} \omega(U, AV) = 0 \text{ for all } U, V \in \mathbb{R}^{p+q+2|2r} \},$$

where $\omega$ is represented by the following matrix $G$:

$$G = \begin{pmatrix} S & 0 \\ 0 & J \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \text{Id}_{p,q} & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \text{Id}_r \\ -\text{Id}_r & 0 \end{pmatrix}, \quad \text{Id}_{p,q} = \begin{pmatrix} \text{Id}_p & 0 \\ 0 & -\text{Id}_q \end{pmatrix}.$$
Equivariant quantization in supergeometry

Fabian Radoux

Orthosymplectic superalgebra of vector fields

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- Bijective correspondence 
  \[ i : C^\infty^{p+q|2r} \to H(\Omega)/H(\Omega) \cap I_F, \text{ where } I_F \text{ is the ideal} \]
  generated by the equation $F$ of the supercone, namely

\[
F(x, \theta) = \sum_{i=2}^{p+1} (x^i)^2 - \sum_{i=p+2}^{p+q+1} (x^i)^2 - 2x^1 x^{p+q+2} + 2 \sum_{i=1}^{r} \theta^i \theta^{i+r}.
\]
Homomorphism $h_{p+q+2,2r}: \mathfrak{osp}(p + 1, q + 1|2r) \rightarrow \text{Vect}(\mathbb{R}^{p+q+2|2r})$. 
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- Homomorphism $h_{p+q+2,2r}$:
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- If $A \in \text{osp}(p + 1, q + 1|2r)$,

\[
\begin{array}{ccc}
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\end{array}
\]

- $\mathfrak{osp}(p+1, q+1|2r)$-equivariant quantization on $\mathbb{R}^{p+q|2r}$: quantization $Q$ s.t. $\mathcal{L}_{X_h} \circ Q = Q \circ L_{X_h}$ for every $h \in \mathfrak{osp}(p+1, q+1|2r)$. 

- Equivariant quantization in supergeometry
- Homomorphism
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Construction of the quantization

- Killing-form of $osp(p + 1, q + 1|2r)$: $K$ given by

$$K : (A, B) \mapsto -\frac{1}{2} \text{str}(AB).$$
Construction of the quantization

- Killing-form of $\mathfrak{osp}(p + 1, q + 1|2r)$: $K$ given by

$$K : (A, B) \mapsto -\frac{1}{2} \text{str}(AB).$$

- Corresponding Casimir operator $C$ on $S^k_{\delta}$:

$$C = \beta_{k,\delta} \text{Id} + R \circ T,$$

where

$$R : S \mapsto i(\omega_0)S, \quad T : S \mapsto \omega_0^\# \lor S,$$

$\omega_0$ bilinear form on $\mathbb{R}^{p+q|2r}$ represented by

$$
\begin{pmatrix}
\text{Id}_{p,q} & 0 \\
0 & J
\end{pmatrix}.
$$
Equivariant quantization in supergeometry

Fabian Radoux

- Eigenvalues of $C$ on $S^k_\delta$: $\alpha_{k,s,\delta}$, $0 \leq s \leq \left\lfloor \frac{k}{2} \right\rfloor$
Equivariant quantization in supergeometry

Fabian Radoux

- Eigenvalues of $C$ on $S^k_\delta$: $\alpha_{k,s,\delta}$, $0 \leq s \leq \lfloor \frac{k}{2} \rfloor$
- If the superdimension $p + q - 2r$ is even and less than or equal to 0, $C$ is not diagonalizable!
Equivariant quantization in supergeometry

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- Multiplicity of $\alpha_{k,s,\delta}$ as root of the minimal polynomial of $C$ is at most two.
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- If the superdimension $p + q - 2r$ is even and less than or equal to 0, $C$ is not diagonalizable!
- Multiplicity of $\alpha_{k,s,\delta}$ as root of the minimal polynomial of $C$ is at most two.
- Quantization is defined on generalized eigenvectors of $C$. 
Case $p + q - 2r \neq 0$:

If $\delta$ is not resonant, then there exists a unique $osp(p + 1, q + 1|2r)$-equivariant quantization.
Case $p + q - 2r \neq 0$:

- If $\delta$ is not resonant, then there exists a unique $\mathfrak{osp}(p + 1, q + 1|2r)$-equivariant quantization.

Proof:

1. If $C$ denotes the Casimir operator on $D_{\lambda, \mu}^k$, for every $S \in \ker(C - \alpha_{k,i,\delta}\text{Id})^2$, $\exists! \hat{S}$ s.t. $\hat{S} \in \ker(C - \alpha_{k,i,\delta}\text{Id})^2$ and s.t. $\sigma_k(\hat{S}) = S$. 
- Case $p + q - 2r \neq 0$:

- If $\delta$ is not resonant, then there exists a unique $\mathfrak{osp}(p + 1, q + 1|2r)$-equivariant quantization.

Proof:

1. If $C$ denotes the Casimir operator on $D^k_{\lambda, \mu}$, for every $S \in \ker(C - \alpha_{k,i}, \delta \text{Id})^2$, $\exists! \hat{S}$ s.t. $\hat{S} \in \ker(C - \alpha_{k,i}, \delta \text{Id})^2$ and s.t. $\sigma_k(\hat{S}) = S$.

2. $Q(S) := \hat{S}$. 
Case $p + q - 2r \neq 0$:

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2. $Q(S) := \hat{\tilde{S}}$.
3. If $S \in \ker(C - \alpha_{k,i,\delta \Id})^2$, $Q(L_{X^h} S) = L_{X^h}(Q(S))$ because they belong to $\ker(C - \alpha_{k,i,\delta \Id})^2$ and because their symbol is $L_{X^h} S$. 
At the order two:

\[ Q = Q_{\text{Aff}} \circ (\text{Id} + a_1 G_0 + a_2 \text{div} + a_3 \Delta_0 + a_4 \text{div}^2), \]

\[ G: S^k_\delta \to S^{k+1}_\delta : S \mapsto \sum_{j=1}^{p+q+2r} (-1)^j \varepsilon^{j\#} \vee \partial_{y_j} S, \]

\[ \Delta: S^k_\delta \to S^k_\delta : S \mapsto \sum_{j=1}^{p+q+2r} \omega_0(e_i, e_j) \partial_{y_j} \partial_{y_i} S, \]

\[ G_0 = G \circ T, \quad \Delta_0 = \Delta \circ T \]
Case $p + q - 2r = 0$:

Arbitrary order: We do not know if we have the existence but the problem does not depend on density weights.
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- Order one:

  $$Q_1 : S \mapsto Q(S) = Q_{\text{Aff}}(S + t \text{div}(S))$$

defines an $\mathfrak{osp}(p + 1, q + 1|2r)$-equivariant quantization for every $t \in \mathbb{R}$ (vector fields in $\mathfrak{osp}(p + 1, q + 1|2r)$ are divergence-free).
Problem setting: find $Q(\nabla) : S(M) \rightarrow D(M)$ such that
Natural and projectively invariant quantizations on supermanifolds (T. Leuther and R.)

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for all superfunction $f$. 
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In a local basis $(\partial_1, \cdots, \partial_{n+m})$ of $\text{Vect}(M)$ ($M$ is of superdimension $(n|m)$), $\Gamma^k_{ij}\partial_k = \nabla_{\partial_i} \partial_j$
Thomas fiber bundle $\tilde{M}$: one adds an even coordinate $x^0$ to each coordinate system $(x^1, \ldots, x^{n+m})$ of $M$. 
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Under a change of coordinates $\tilde{x}(x)$, $x^0$ transforms into $x^0 + \log |\text{Ber}A|$, where

$$A^i_j = \frac{\partial \tilde{x}^j}{\partial x^i}$$
Thomas connection (J. George)

\[ \nabla \text{ and } \nabla' \text{ are projectively equivalent iff } \Pi^k_{ij} = \Pi'^k_{ij}, \text{ where} \\
\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n - m + 1} (\Gamma^s_{is} \delta^k_j (-1)^{\tilde{s}} + \Gamma^s_{js} \delta^k_i (-1)^{\tilde{i}j + \tilde{s}}) \]
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- Thomas connection $\tilde{\nabla}$ on $\tilde{M}$:

$$\tilde{\Gamma}_{ij}^k = \Pi_{ij}^k, \quad \tilde{\Gamma}_c^0 = \tilde{\Gamma}_a^0 = \frac{-\delta_c^a}{n-m+1},$$

$$\tilde{\Gamma}_i^0 = \frac{n-m+1}{n-m-1} \left( \partial_q \Pi_{ij}^q - \Pi_{qi}^p \Pi_{pj}^q \right) (-1)^{\tilde{q}(\tilde{a} + \tilde{i} + \tilde{j})}$$
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\[ \text{Thomas connection } \tilde{\nabla} \text{ on } \tilde{M}: \]

\[ \tilde{\Gamma}_{ij}^k = \Pi_{ij}^k, \quad \tilde{\Gamma}_{0a}^c = \tilde{\Gamma}_{a0}^c = \frac{-\delta_a^c}{n-m+1}, \]

\[ \tilde{\Gamma}_{ij}^0 = \frac{n-m+1}{n-m-1} \left( \partial_q \Pi_{ij}^q - \Pi_{qi}^p \Pi_{pj}^q \right) (-1)^{q(\tilde{q}+\tilde{i}+\tilde{j})} \]

\[ \tilde{\nabla} \text{ depends on } \nabla \text{ in a natural and projectively invariant way; moreover, } L_\mathcal{E} \tilde{\nabla} = 0 \text{ with } \mathcal{E} = \partial_0 \]
Construction of the quantization

- Bijective correspondence $i$ between $\lambda$-densities on $M$ and $\lambda$-equivariant functions on $\tilde{M}$:
  
  $$i : f \mapsto \tilde{f}, \quad L\mathcal{E}\tilde{f} = \lambda\tilde{f}$$
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Construction of the quantization

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- Natural and projectively invariant lift of symbols: $S \mapsto \tilde{S}$ with $L_\xi \tilde{S} = \delta \tilde{S}$.

- Natural canonical quantization $\tau$: if $S$ is a symbol of degree $k$, then
  \[ \tau(\nabla)(S)(f) := \langle S, \nabla^k f \rangle \]
Equivariant quantization in supergeometry

\[ \tilde{f} \xrightarrow{\tau(\tilde{\nabla})(\tilde{S})} \tau(\tilde{\nabla})(\tilde{S})(\tilde{f}) \]

\[ f \xrightarrow{Q(\nabla)(S)} Q(\nabla)(S)(f) \]

Case \( n-m=1 \): Thomas connection not defined. For \( m=0 \), there is no quantization.

\[ \rightarrow \text{Conjecture of the non-existence of the quantization} \]

Case \( n-m=-1 \): Thomas connection not defined. But the quantization exists at order two and \( pgl(n+1,n+1) \)-equivariant quantization on \( \mathbb{R}^n|_{n+1} \) exists.

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Fabian Radoux

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