Equivariant quantization in supergeometry

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- The set of quantizable observables: $\mathrm{Pol}_{\leqslant 1}(T^*M)$.

- Geometric quantization Q_G: $Q_G = Q|_{\mathrm{Pol}_{\leqslant 1}(T^*M)}$,

$$Q_G(X^i(x)p_i + A(x)) = \frac{\hbar}{i} X^i(x)\partial_i + A(x).$$

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- Is it possible to extend the geometric quantization to $\mathrm{Pol}(T^*M) \cong \mathcal{S}(M)$?
- Is this prolongation unique?
- Is it possible to reestablish the uniqueness?

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$$Q_{\mathrm{aff}}(S^{i_1 \cdots i_k} \partial_{i_1} \vee \cdots \vee \partial_{i_k}) = S^{i_1 \cdots i_k} \partial_{i_1} \cdots \partial_{i_k}$$

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$$\begin{aligned} S^{i_1\cdots i_k}\partial_{i_1} \vee \cdots \vee \partial_{i_k} &= S^{j_1\cdots j_k} J^{i_1}_{j_1}\cdots J^{i_k}_{j_k}\bar{\partial}_{i_1} \vee \cdots \vee \bar{\partial}_{i_k} \\ S^{i_1\cdots i_k}\partial_{i_1}\cdots\partial_{i_k} &= S^{j_1\cdots j_k} J^{i_1}_{j_1}\cdots J^{i_k}_{j_k}\bar{\partial}_{i_1}\cdots\bar{\partial}_{i_k} + \cdots \end{aligned}$$

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- $L_X Q(\nabla_0)(S) = Q(\nabla_0)(L_X S)$ for all $X \in \mathfrak{sl}(m+1, \mathbb{R})$

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- $Q\widetilde{(\nabla)(S)}(f) = \tau(\tilde{\nabla})(\tilde{S})(\tilde{f})$ with τ a canonical natural quantization

Aims of the talk

Show how to superize and to solve in the super setting the following problems:

- Projectively equivariant quantization on $\mathbb{R}^m$
- Conformally equivariant quantization on $\mathbb{R}^m$
- Natural and projectively invariant quantization

Projectively equivariant quantization on $\mathbb{R}^{n|m}$ (P. Mathonet, R.)

- Superfunction f on a supermanifold of dimension $(n|m)$: locally, $f(x^1, \ldots, x^n) = \sum_{I \subseteq \{1,\ldots,m\}} f_I(x^1, \ldots, x^n)\theta^I$, $\theta^i\theta^j = -\theta^j\theta^i$.

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- Super vector field: superderivation of the superalgebra of superfunctions.
- Locally, a λ-density is expressed formally as $f|Dx|^\lambda$. Under a change of coordinates $\bar{x}(x)$, $|Dx|^\lambda$ is multiplied by $|\mathrm{Ber} A|^\lambda$, with $A^i_j = \frac{\partial \bar{x}^j}{\partial x^i}$.

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- Superfunction f on a supermanifold of dimension $(n|m)$: locally, $f(x^1,\ldots,x^n) = \sum_{I\subseteq\{1,\ldots,m\}} f_I(x^1,\ldots,x^n)\theta^I$, $\theta^i\theta^j = -\theta^j\theta^i$.
- Super vector field: superderivation of the superalgebra of superfunctions.
- Locally, a λ-density is expressed formally as $f|Dx|^\lambda$. Under a change of coordinates $\bar{x}(x)$, $|Dx|^\lambda$ is multiplied by $|\mathrm{Ber} A|^\lambda$, with $A^i_j = \frac{\partial \bar{x}^j}{\partial x^i}$.
  Moreover,

$$L_X(f|Dx|^\lambda) = (X(f) + \lambda\,\mathrm{div}(X) f)|Dx|^\lambda,$$

  where

$$\mathrm{div}(X) = \sum_{i=1}^{n+m} (-1)^{\tilde{y}_i\widetilde{X^i}} \partial_{y^i} X^i.$$

- Differential operator $D \in \mathcal{D}^k_{\lambda,\mu}$:

$$D = \sum_{|\alpha|\leqslant k} D_\alpha \partial_{x^1}^{\alpha_1} \cdots \partial_{x^n}^{\alpha_n} \partial_{\theta^1}^{\alpha_{n+1}} \cdots \partial_{\theta^m}^{\alpha_{n+m}},$$

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$$\sigma_k : \mathcal{D}^k \to \mathcal{S}^k : D \mapsto \sum_{|\alpha| = k} D_\alpha \otimes \partial_1^{\alpha_1} \vee \cdots \vee \partial_{n+m}^{\alpha_{n+m}}.$$

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- Quantization: linear bijection $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ s.t. $\sigma_k(Q(S)) = S$ for all $S \in \mathcal{S}^k_\delta$.

Projective superalgebra of vector fields on $\mathbb{R}^{n|m}$

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- If $A \in \mathfrak{gl}(n+1|m)$,

$$\begin{array}{ccc} H(\Omega) & \xrightarrow{h_{n+1,m}(A)} & H(\Omega) \\ \uparrow i & & \downarrow i^{-1} \\ C^{\infty n|m} & \xrightarrow{\pi(h_{n+1,m}(A))} & C^{\infty n|m} \end{array}$$

- $\pi \circ h_{n+1,m}(\mathrm{Id}) = 0$, thus $\pi \circ h_{n+1,m}$ induces a homomorphism from $\mathfrak{pgl}(n+1|m)$ to $\mathrm{Vect}(\mathbb{R}^{n|m})$.

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- Casimir operator of (V, β):

$$\sum_{i=1}^{n}(-1)^{\tilde{u}_i}\beta(u_i)\beta(u'_i) = \sum_{i=1}^{n}\beta(u'_i)\beta(u_i).$$

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- The Casimir operator C of $\mathfrak{pgl}(n+1|m) \cong \mathfrak{sl}(n+1|m)$ on $(\mathcal{S}^k_\delta, L)$ is equal to $\alpha(k,\delta)\mathrm{Id}$.

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  2. $Q(S) := \hat{S}$.
  3. If $S \in \mathcal{S}^k_\delta$, $Q(L_{X^h}S) = \mathcal{L}_{X^h}(Q(S))$ because they are eigenvectors of $\mathcal{C}$ of eigenvalue $\alpha(k,\delta)$ and because their symbol is $L_{X^h}S$.

Divergence operator:

$$\mathrm{div} : \mathcal{S}^k_\delta \to \mathcal{S}^{k-1}_\delta : S \mapsto \sum_{j=1}^{n+m} (-1)^{\tilde{y^j}} i(dx^j)\partial_{y^j} S.$$

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Theorem

If δ is not critical, then the map $Q : \mathcal{S}_\delta \to \mathcal{D}_{\lambda,\mu}$ defined by

$$Q(S)(f) = \sum_{r=0}^{k} C_{k,r}\, Q_{\mathrm{Aff}}(\mathrm{div}^r S)(f), \quad \textit{for all } S \in \mathcal{S}^k_\delta$$

is the unique $\mathfrak{sl}(n+1|m)$-equivariant quantization if

$$C_{k,r} = \frac{\prod_{j=1}^{r}((n-m+1)\lambda + k - j)}{r!\, \prod_{j=1}^{r}(n-m+2k-j-(n-m+1)\delta)}\ \forall r \geqslant 1.$$

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is a nondegenerate invariant supersymmetric even form, we can then apply the Casimir operator method to $\mathfrak{psl}(n+1|n+1)$.
- If $k \neq 1$, Q is given by the same formula as in the case $m \neq n+1$.

- If $k = 1$,

$$Q_1 : S \mapsto Q(S) = Q_{\mathrm{Aff}}(S + t\,\mathrm{div}(S))$$

defines a $\mathfrak{psl}(n+1|n+1)$-equivariant quantization for every $t \in \mathbb{R}$ (vector fields in $\mathfrak{psl}(n+1|n+1)$ are divergence-free).

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- The $\mathfrak{psl}(n+1, n+1)$-equivariant quantizations are $\mathfrak{pgl}(n+1, n+1)$-equivariant (equivariance with respect to the Euler vector field).
- Q does not depend on δ and λ.

Orthosymplectically equivariant quantizations on $\mathbb{R}^{n|2r}$ (T. Leuther, P. Mathonet, R.)

- $\mathfrak{osp}(p+1, q+1|2r)$:

$$\{A \in \mathfrak{gl}(p+q+2|2r) : \omega(AU, V) + (-1)^{\tilde{A}\tilde{U}}\omega(U, AV) = 0 \text{ for all } U, V \in \mathbb{R}^{p+q+2|2r}\},$$

where ω is represented by the following matrix G:

$$G = \begin{pmatrix} S & 0 \\ 0 & J \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \mathrm{Id}_{p,q} & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & \mathrm{Id}_r \\ -\mathrm{Id}_r & 0 \end{pmatrix}, \mathrm{Id}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}.$$

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- Bijective correspondence $i : C^{\infty p+q|2r} \to H(\Omega)/H(\Omega) \cap I_F$, where I_F is the ideal generated by the equation F of the supercone, namely

$$F(x, \theta) = \sum_{i=2}^{p+1} (x^i)^2 - \sum_{i=p+2}^{p+q+1} (x^i)^2 - 2x^1 x^{p+q+2} + 2 \sum_{i=1}^{r} \theta^i \theta^{i+r}.$$

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- Corresponding Casimir operator C on $\mathcal{S}^k_\delta$:

$$C=\beta_{k,\delta}\mathrm{Id}+R\circ T,$$

$$R:S\mapsto i(\omega_0)S,\quad T:S\mapsto\omega_0^\sharp\vee S,$$

ω_0 bilinear form on $\mathbb{R}^{p+q|2r}$ represented by

$$\begin{pmatrix}\mathrm{Id}_{p,q} & 0\\ 0 & J\end{pmatrix}.$$

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  3. If $S \in \ker(C - \alpha_{k,i,\delta}\mathrm{Id})^2$, $Q(L_{X^h} S) = \mathcal{L}_{X^h}(Q(S))$ because they belong to $\ker(\mathcal{C} - \alpha_{k,i,\delta}\mathrm{Id})^2$ and because their symbol is $L_{X^h} S$.

- At the order two:

$$Q = Q_{\mathrm{Aff}} \circ (\mathrm{Id} + a_1 G_0 + a_2 \mathrm{div} + a_3 \Delta_0 + a_4 \mathrm{div}^2),$$

$$G : \mathcal{S}^k_\delta \to \mathcal{S}^{k+1}_\delta : S \mapsto \sum_{j=1}^{p+q+2r} (-1)^{\tilde{j}} \varepsilon^{j\sharp} \vee \partial_{y^j} S,$$

$$\Delta : \mathcal{S}^k_\delta \to \mathcal{S}^k_\delta : S \mapsto \sum_{j=1}^{p+q+2r} \omega_0(e_i, e_j) \partial_{y^j} \partial_{y^i} S,$$

$$G_0 = G \circ T, \quad \Delta_0 = \Delta \circ T$$

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- Order one:

$$Q_1 : S \mapsto Q(S) = Q_{\mathrm{Aff}}(S + t\,\mathrm{div}(S))$$

defines an $\mathfrak{osp}(p+1, q+1|2r)$-equivariant quantization for every $t \in \mathbb{R}$ (vector fields in $\mathfrak{osp}(p+1, q+1|2r)$ are divergence-free).

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- In a local basis $(\partial_1, \cdots, \partial_{n+m})$ of $\mathrm{Vect}(M)$ (M is of superdimension $(n|m)$), $\Gamma_{ij}^k \partial_k = \nabla_{\partial_i} \partial_j$

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$$A^i_j = \frac{\partial \bar{x}^j}{\partial x^i}$$

Thomas connection (J. George)

- ∇ and ∇' are projectively equivalent iff $\Pi_{ij}^k = \Pi'^k_{ij}$, where

$$\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{n-m+1}(\Gamma_{is}^s \delta_j^k (-1)^{\tilde{s}} + \Gamma_{js}^s \delta_i^k (-1)^{\tilde{i}\tilde{j}+\tilde{s}})$$

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- $\tilde{\nabla}$ depends on ∇ in a natural and projectively invariant way; moreover, $\mathrm{L}_{\mathcal{E}}\tilde{\nabla} = 0$ with $\mathcal{E} = \partial_0$

Construction of the quantization

- Bijective correspondance i between λ-densities on M and λ-equivariant functions on $\tilde{M}$:

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- Natural canonical quantization τ: if S is a symbol of degree k, then

$$\tau(\nabla)(S)(f) := \langle S, \nabla^k f\rangle$$

- $$\begin{array}{ccc} \tilde{f} & \xrightarrow{\tau(\tilde{\nabla})(\tilde{S})} & \tau(\tilde{\nabla})(\tilde{S})(\tilde{f}) \\ \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle i^{-1}} \\ f & \xrightarrow{Q(\nabla)(S)} & Q(\nabla)(S)(f) \end{array}$$

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- Case n-m=-1: Thomas connection not defined. But the quantization exists at order two and $\mathfrak{pgl}(n+1, n+1)$-equivariant quantization on $\mathbb{R}^{n|n+1}$ exists
  $\longrightarrow$ Conjecture of the existence of the quantization